Exercises

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1. Let $\phi_i : U_i \to V_i$, $i = 1, 2$ be two complex charts on a topological space X with $U_1 \cap U_2 \neq \varnothing$. Assume $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is holomorphic. Show that $\phi_2 \circ \phi_1^{-1}$ is a biholomorphic map.

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2. (a) Let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ and τ the usual topology of $\mathbb{C}$. Define the topology τ_∞ on $\mathbb{C}_\infty$ by taking the union of τ with the collection of sets of the form $V \cup \{\infty\}$, where V is the complement of a compact subset of $\mathbb{C}$. Show that $\mathbb{C}$ is not compact, but $\mathbb{C}_\infty$ is compact.
   (b) Let $\mathbb{P}^1$ be the set of complex one-dimensional subspaces of $\mathbb{C}^2$. Denote by $[v : w]$ the subspace of $\mathbb{C}^2$ generated by the non-zero element (v, w) of $\mathbb{C}^2$. Show that the map $\mathbb{P}^1 \to \mathbb{C}_\infty$ that sends $[v : w]$, $w \neq 0$ to v/w and $[v : 0]$ to ∞ is a bijection.
   (c) Let $S^2 = \{(x, y, w) \in \mathbb{R}^3 | x^2 + y^2 + w^2 = 1\}$ be the two-sphere in $\mathbb{R}^3$. Check that the map $\phi : S^2 \to \mathbb{C}_\infty$ that sends $(x, y, w) \neq (0, 0, 1)$ to $\frac{x+iy}{1-w}$, and $(0, 0, 1)$ to ∞ is a bijection.
   (d) Define a complex atlas for each of the previous examples. Show that the bijections are biholomorphic maps.

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3. Let X be a Riemann surface. Show that the set $\mathcal{O}(X)$ of holomorphic functions on X is a $\mathbb{C}$-algebra. Conclude that this is also true for any open subset U of X (consider the connected components of U).

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4. Let f be a holomorphic map between a Riemann surface X and $\mathbb{C}$ seen as a Riemann surface. Show that f is a holomorphic map in the usual sense.

5. Let $f : X \to Y$ and $g : Y \to Z$ be holomorphic maps between Riemann surfaces. Show that $g \circ f : X \to Z$ is a holomorphic map.

6. Let $f : X \to Y$ a continuous map between Riemann surfaces. Let the *pull back* of a function ϕ on V open subset of Y be the function $f^*(\phi)$ on $f^{-1}(V)$ given by the composition $\phi \circ f$.
   (a) Show that f is holomorphic iff for all open subsets V in Y and all $\phi \in \mathcal{O}(V)$ we have that $f^*(\phi) \in \mathcal{O}(f^{-1}(V))$.
   (b) Show that if f is holomorphic, f^* is a homomorphism of $\mathbb{C}$-algebras.

7. Let $f : X \to Y$ and $g : Y \to Z$ be holomorphic maps between Riemann surfaces. Show that $(g \circ f)^* = f^* \circ g^*$.

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8. Show that a holomorphic function $f : X \to Y$, X, Y Riemann surfaces, X compact can assume a value $q \in Y$ only a finite number of times.

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9. Show that a meromorphic function on a compact Riemann surface has a finite number of zeros and poles.

10. Show that any rational function, i.e., the ratio of two polynomials $P(z)/Q(z)$, $Q \not\equiv 0$, defines a meromorphic function on the Riemann sphere.

11. Show that if f is a meromorphic function on a Riemann surface X, $g \not\equiv 0$, then $1/f \in \mathcal{M}(X)$, by using Riemann's removable singularity theorem.

12. Let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Show that the *linear fractional transformation*

$$f(z) = \frac{az + b}{cz + d},$$

defines a meromorphic function on $\mathbb{P}^1$. Show that the corresponding map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic.

13. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a biholomorphic map. Let z be the usual coordinate on $\mathbb{C} \subset \mathbb{P}^1$. Show that $f(z)$ is given by a linear fractional transformation.

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14. Let f be a rational function seen as a meromorphic map on the Riemann sphere. Show that f can be written as

$$f(z) = c \prod_i (z - z_i)^{e_i}$$

for some integers e_i and distinct z_i. Show that if $e_i > 0$ then z_i is a zero of order[1] e_i and if $e_i < 0$, then z_i is a pole of order $-e_i$. Compute the order of the pole or zero at ∞. Show that the sum of all orders of zeros and all orders of poles is zero.

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15. Let X be a compact Riemann surface and $f \in \mathcal{M}(X)$, f non-constant. Show that f must have at least a zero and a pole on X.

[1] Recall that the *order* of a meromorphic function f on the Riemann surface X at a pole or at a zero $p \in X$ is given by the valency at p of f seen as an holomorphic map between X and $\mathbb{P}^1$.

16. Let Γ be the lattice in $\mathbb{C}$ generated by w_1, w_2 complex numbers linearly independent over $\mathbb{R}$, i.e.,

$$\Gamma = w_1\mathbb{Z} + w_2\mathbb{Z}.$$

Let $\alpha \in \mathbb{C}$ non-zero and Γ' the lattice defined by

$$\Gamma' = \alpha w_1\mathbb{Z} + \alpha w_2\mathbb{Z}.$$

Show that the map $\phi : \mathbb{C} \to \mathbb{C}$ which multiplies a complex number by α descends to a biholomorphic map between the associated tori:

$$\phi : \frac{\mathbb{C}}{\Gamma} \to \frac{\mathbb{C}}{\Gamma'}.$$

17. Show that any complex torus $\frac{\mathbb{C}}{\Gamma}$ is isomorphic to a complex torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$, where τ is a complex number with strictly positive imaginary part.

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18. Show that a holomorphic map between compact Riemann surfaces is an isomorphism iff it has degree one.

19. Let $f \in \mathcal{M}(X)$ non-constant, X a compact Riemann surface. Show that

$$\sum_{p \in X} \operatorname{ord}_f(p) = 0.$$

20. Let $f : X \to Y$ and $g : Y \to Z$ be two non-constant holomorphic maps between compact Riemann surfaces. Show that, for $p \in X$, $\operatorname{val}_{g\circ f}(p) = \operatorname{val}_f(p)\operatorname{val}_g(f(p))$.

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21. Let $C \subset \mathbb{C}^2$ be a plane algebraic curve defined by a polynomial $f(z, w)$. Suppose at the points $p_1, \ldots, p_n \in C$ are singular, i.e., $(\partial f/\partial z, \partial f/\partial w) = (0, 0)$ at each p_i. Show that $C/\{p_1, \ldots, p_n\}$ is a Riemann surface.

22. Show that the algebraic curve with equation $w^2 = h(z)$ is smooth if and only if the polynomial $h(z)$ as distinct roots.

23. Let C be an plane algebraic curve defined by a polynomial $f(z, w)$ of degree 2, i.e., it is an *affine conic*. Suppose C has a singular point. For simplicity let $f(z, w) = w^2 - h(z)$. Show that the polynomial f factors as the product of two linear polynomials, so C is the union of two lines.

24. Let C be a smooth place algebraic curve in $\mathbb{C}^2$ defined by $f(z, w) = 0$. Show that the map $\pi : C \to \mathbb{C}$ defined by $\pi(z, w) = z$ is a holomorphic map between Riemann surfaces. Show that $p \in C$ is a ramification point, i.e. $\operatorname{val}_\pi(p) > 1$, iff $\partial f/\partial w$ vanishes at p.

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25. (a) By drawing pictures convince yourself that gluing the corresponding sides of the 8-gon
$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$$
one gets a sphere S with 2 handles, i.e., a 2-manifold of genus 2.

    (b) By dividing the 8-gon in triangles compute the Euler number of S and check for this example the formula relating the Euler number and the genus of a surface.

    (c) Produce a different triangulation of the same 8-gon and show that Euler number does not change.

26. Produce two different triangulations T and T' of the 8-gon. Superimpose the two triangulations make it a triangulation T'' by adding vertices and edges. Check that T'' is a refinement of both T and T' (show that both can be made into T'' by sequences of elementary refinements).

27. Show that a biholomorphic map between open subsets of the complex plane preserves the orientation.

28. Let
$$f(z) = z^3/(1 - z^2)$$
define a meromorphic function on the Riemann sphere. Find all points p such that $\mathrm{ord}_p(f) \neq 0$. Find the degree of f, as map from $\mathbb{P}^1$ to $\mathbb{P}^1$. Find all ramification points and branch points. Verify Riemann-Hurwitz formula.

29. Same as above for the meromorphic function
$$f(z) = 4z^2 \frac{(z-1)^2}{(2z-1)^2}.$$

30. Let $f : X \to Y$ a non-constant holomorphic map between compact Riemann surfaces.

    (a) Show that if $Y \simeq \mathbb{P}^1$ and $\deg f > 1$, then f is ramified.

    (b) Show that if both X and Y have genus one, then f is unramified.

    (c) Show that g_X is always bigger than or equal to g_Y.

    (d) Show that if $g_X = g_Y \geqslant 2$, then f is an isomorphism.