MaPC4A - CT - 5/5/2023

① a) $\sin\left(\frac{\pi}{2}+i\right) = \frac{1}{2i}\left(e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}\right) =$

$$= \frac{1}{2i}\left(e^{-1}i + ei\right) = \frac{1}{2}\left(e + \frac{1}{e}\right)$$

$$\mathrm{Re}\left[\sin\left(\frac{\pi}{2}+i\right)\right] = \frac{1}{2}\left(e + \frac{1}{e}\right)$$

$$\mathrm{Im}\left[\sin\left(\frac{\pi}{2}+i\right)\right] = 0$$

b) $z^5 = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}} = 2^{1/2} e^{i\frac{\pi}{4}+2\pi i k} \qquad k \in \mathbb{Z}$

$$\Rightarrow \quad z_k = 2^{1/10} e^{i\frac{\pi}{20} + \frac{2}{5}\pi i k} \qquad k = 0, \dots, 4$$

$$|z_k| = 2^{1/10}$$

$$\arg z_k = \frac{\pi}{20}(1+8k) \mod 2\pi \qquad k = 0, \dots, 4$$

$$= \frac{\pi}{20}, \frac{9\pi}{20}, \frac{17\pi}{20}, \frac{25\pi}{20}, \frac{33\pi}{20} \mod 2\pi$$

$$\mathrm{Arg}\, z_k = \frac{\pi}{20}, \frac{9\pi}{20}, \frac{17\pi}{20}, -\frac{15\pi}{20}, -\frac{7\pi}{20}$$

② $a_n = n^k \qquad R = \lim\limits_{n\to+\infty} \frac{n^k}{(n+1)^k} = 1$

$\Rightarrow$ Le domaine de convergence est $D(0,1)$.

③ a) $u = 3e^{-2y} \cos 2x$

CR : $u_x = v_y \Rightarrow v_y = -2\cdot 3\, e^{-2y} \sin 2x$

$\Rightarrow v = 3e^{-2y} \sin 2x + s(x)$

CR: $u_y = -v_x$

$\Rightarrow -2\cdot 3\, e^{-2y} \cos 2x = -2\cdot 3\, e^{-2y} \cos 2x - s'(x)$

$\Rightarrow s(x) = c$

$\Rightarrow v = 3e^{-2y} \sin 2x + c \qquad g(0) = 3 \Rightarrow c = 0$

b) $g = 3e^{-2y}(\cos 2x + i \sin 2x)$

$\overset{!}{=} 3e^{-2y} e^{i2x}$

$\overset{!}{=} 3e^{i2(x+iy)}$

$\overset{!}{=} 3e^{2iz}$

④ a) Pôles : 0 et 1 d'ordre 1

b) $f(z) = -\frac{2}{z} + \frac{2}{z-1}$

c) Pour $|z| < 1$ on a $\frac{2}{z-1} = \frac{-2}{1-z} = -2 \sum_{k \geq 0} z^k$,

donc : $f(z) = -2 \sum_{k \geq -1} z^k = -2z^{-1} - 2 - 2z + \cdots$

pour $0 < |z| < 1$.

d) Pour $|z|>1$ on a $\frac{2}{z-1} = \frac{2}{z}\frac{1}{1-\frac{1}{z}} = 2\sum_{k\geq 0} z^{-k-1}$

donc : $f(z) = -\frac{2}{z} + 2\sum_{k\geq 0} z^{-k-1} = 2\sum_{k\geq 1} z^{-k-1} = 2z^{-2} + \cdots$

pour $|z|>1$.

e) $\underset{z=0}{\text{Rés}}\, f(z) = -2$ $\qquad \underset{z=1}{\text{Rés}}\, f(z) = 2$

f) $\int_C f(z)\,dz = 2\pi i \underset{z=0}{\text{Rés}}\, f(z) = -4\pi i$

g) $\int_{C'} f(z)\,dz = 2\pi i \underset{z=0}{\text{Rés}}\, f(z) + 2\pi i \underset{z=1}{\text{Rés}}\, f(z) = 0$

5 a) $\text{Re}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}\,dx = \int_{-\infty}^{+\infty} \frac{\text{Re}\, e^{ix}}{x^2+1}\,dx = I$

b) La fonction $h(z) = \frac{e^{iz}}{z^2+1}$ a deux pôles $z = \pm i$.

$\int_C h(z)\,dz = 2\pi i \underset{z=i}{\text{Rés}}\, h(z) = 2\pi i \cdot \frac{1}{2ei} = \frac{\pi}{e}$

c) $z(t) = Re^{it}$ $\qquad t \in [0, \pi]$

$$\int_C h(z)\,dz = \int_0^{\pi} h(Re^{it})\, Rie^{it}\,dt =$$

$$= \int_0^{\pi} \frac{e^{iRe^{it}}}{R^2e^{2it}+1}\, Rie^{it}\,dt$$

d) $$\left|\int_C h(z)\, dz\right| \leqslant \int_0^{\pi} \left|\frac{e^{iRe^{it}}}{R^2e^{2it}+1}\, Rie^{it}\right| dt = $$

$$= R\int_0^{\pi} \frac{\left|e^{iRe^{it}}\right|}{\left|R^2e^{2it}+1\right|}\, dt$$

$$\left|e^{iRe^{it}}\right| = e^{\mathrm{Re}(iRe^{it})} = e^{-R\sin t} \xrightarrow[R\to+\infty]{} 0$$

puisque $\sin t > 0$ pour $t \in (0, \pi)$.

e) $$\int_{\mathbb{R}} h(z)\, dz = \lim_{R\to+\infty} \int_C h(z)\, dz = \frac{\pi}{e}$$