Partial Differential Equations
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Contents

1 Introduction: ODEs and PDEs

Most theories and models that describe quantitatively our world are based on differential equations.

Differential equations are equations that contain the derivatives of an unknown function y with respect to one or more independent variables.

We distinguish between ordinary differential equations (ODEs) and partial differential equations (PDEs).

GAP 1

1.1 Examples of ODEs and PDEs in physics, chemistry, biology, economics, ...

1.1.1 ODEs

Important examples of first order ODEs:

- Law of radioactive decay

  GAP 2

- Reaction speed equation (for a second order reactant)

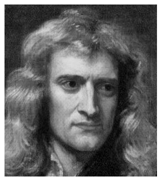
  GAP 3

- Torricelli's law

E. Torricelli

GAP 4

- Newton's law of cooling



I. Newton

GAP 5

- Malthus (1798) law of population dynamics, i.e., exponential behaviour

T. R. Malthus

GAP 6

- Logistic equation (P. Verhulst, 1845), the simplest population growth model with bounded growth

GAP 7

Exercise 1. Check that the general solution to the logistic equation is given by

$$P(t) = \frac{r}{k + (r/P_0 - k)e^{-rt}} \tag{1}$$

and observe that in the limit $t \to +\infty$ the the population approaches the maximum sustainable population r/k.

In all these examples *physical laws* are translated in *differential equations*. By solving the related *initial value problem* one can make *predictions*.

The initial value problem for a first order ODE

GAP 8

in general determines the solution $y(x)$ uniquely.

Up to 1950 the logistic model has a remarkable predictive power of US population:

Year	Actual U.S. Pop.	Exponential Model	Exponential Error	Logistic Model	Logistic Error
1800	5.308	5.308	0.000	5.308	0.000
1810	7.240	6.929	0.311	7.202	0.038
1820	9.638	9.044	0.594	9.735	−0.097
1830	12.861	11.805	1.056	13.095	−0.234
1840	17.064	15.409	1.655	17.501	−0.437
1850	23.192	20.113	3.079	23.192	0.000
1860	31.443	26.253	5.190	30.405	1.038
1870	38.558	34.268	4.290	39.326	−0.768
1880	50.189	44.730	5.459	50.034	0.155
1890	62.980	58.387	4.593	62.435	0.545
1900	76.212	76.212	0.000	76.213	−0.001
1910	92.228	99.479	−7.251	90.834	1.394
1920	106.022	129.849	−23.827	105.612	0.410
1930	123.203	169.492	−46.289	119.834	3.369
1940	132.165	221.237	−89.072	132.886	−0.721
1950	151.326	288.780	−137.454	144.354	6.972
1960	179.323	376.943	−197.620	154.052	25.271
1970	203.302	492.023	−288.721	161.990	41.312
1980	226.542	642.236	−415.694	168.316	58.226
1990	248.710	838.308	−589.598	173.252	76.458
2000	281.422	1094.240	−812.818	177.038	104.384

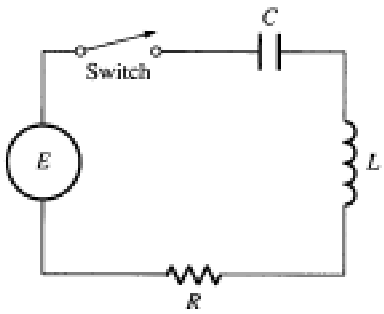
FIGURE 1.7.4. Comparison of exponential growth and logistic models with U.S. census populations (in millions).

Second order ODEs include most examples from mechanics, because of Newton's law $F = ma$. Examples:

- Spring + mass + dashpot (shock absorber) system

GAP 9

- The series RLC circuit



GAP 10

1.1.2 PDEs

Examples of important PDEs:

- Wave equation

GAP 11

- Maxwell's equations (electrodynamics, optics, electric circuits, ...)

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \tag{2}$$

$$\nabla \cdot B = 0 \tag{3}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{4}$$

$$\nabla \times B = \mu_0 (J + \epsilon_0 \frac{\partial E}{\partial t}). \tag{5}$$

James Clerk Maxwell

GAP 12

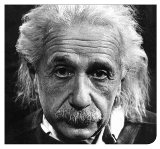
In free space ($\rho = 0 = J$), these imply that the components of E and B satisfy the *wave equation*:

GAP 13

Hence the propagating speed of electromagnetic radiation (i.e. light) is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sim 3 \cdot 10^8 m/s. \tag{6}$$

- Einstein field equations (general relativity): ten coupled nonlinear PDEs for the metric tensor $g_{\alpha\beta}$ (describing the geometry of space-time) in terms of the stress-energy tensor $T_{\alpha\beta}$ (describing the matter/energy content of space-time).



Albert Einstein

GAP 14

- Schrödinger's equation (quantum mechanics): linear PDE describing the evolution in time of the quantum state (given by the *wavefunction* ψ) of a particle of mass m in an electric field with potential V:

Erwin Schrödinger

GAP 15

- Diffusion (or heat) equation (thermodynamics, diffusion processes)

GAP 16

- Laplace equation: steady states (i.e., time-independent) of the heat equation

GAP 17

- An example of PDE in chemistry: the reaction-diffusion equation models the concentration of substances distributed in space under the effect of chemical reactions and diffusion:

GAP 18

- In finance the Black-Scholes equation (1973) models the price of a derivative V as a function of the stock price S and time t:

GAP 19

- Korteweg - de Vries equation (KdV) models shallow water nonlinear waves. It is exactly solvable!

GAP 20

2 Ordinary differential equations

2.1 Basic concepts

A *differential equation* is an equation involving an unknown function y and its derivatives with respect to independent variables usually denoted $x, t, \ldots$.

If the unknown function depends on only one independent variable we have an *ordinary differential equation.* If the unknown function depends on more than one independent variable then we have a *partial differential equation.* We further distinguish between *linear* and *nonlinear* differential equations.

The *order* of a differential equation is the order of the highest derivative appearing in it.

Example 1. The following are differential equations for the unknown function y. Which ones are ODEs and which PDEs? What is their order? Which are linear and which – nonlinear? What are the independent variables ?

$$\frac{dy}{dx} = 5x + 3 \tag{1}$$

GAP 1

$$e^y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1 \tag{2}$$

GAP 2

$$\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 1 \tag{3}$$

GAP 3

$$\frac{\partial y}{\partial t} - k\frac{\partial^2 y}{\partial x^2} = 0 \tag{4}$$

GAP 4

$$\frac{\partial y}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 y}{\partial s^2} + rs\frac{\partial y}{\partial s} + ry = 0 \tag{5}$$

GAP 5

2.2 First order differential equations

A first order ordinary differential equation can be written (in standard form) as

$$y' = f(x, y). \tag{6}$$

A *solution* is a function $y(x)$ defined on an interval $I \subset \mathbb{R}$ such that when substituted in (6), it becomes an identity for each $x \in I$. In general one has infinitely many solutions to equation (6), depending on a constant C. Such family is called *general solution*.

If the function f depends only on x the problem of finding the solution of (6) becomes the problem of finding the primitive (antiderivative) of $f(x)$.

GAP 6

In some cases the solution can be found explicitly: consider the equation $y' = ay$ for a constant a

GAP 7

Solutions may not be defined on the whole x-axis: consider the equation $y' = y^2$

GAP 8

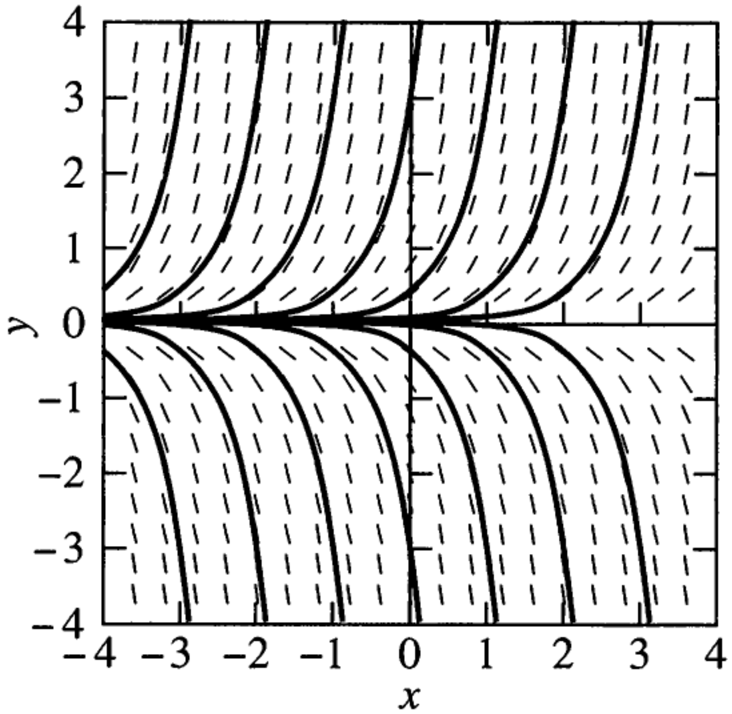
Real solutions might not exist at all: consider the equation $(y')^2 + y^2 = -1$

GAP 9

We can interpret geometrically equation (6) as a slope field or direction field

GAP 10

For example, for the equation $y' = 2y$ we have:



Given an equation of the type (6), one faces two problems: 1) the existence and uniqueness (under certain extra assumptions) of the solutions, 2) finding such solutions and establishing their properties.

Let's first consider the problem of existence and uniqueness of solutions.

2.2.1 The problem of existence and uniqueness

In general we expect on ODE to have an infinite family of solutions. To specify uniquely a solution we thus need additional requirements, for example *initial conditions*.

The *initial value problem* or *Cauchy problem* is the following:

GAP 11

Let R be a rectangle containing the point (x_0, y_0) e.g.

$$R = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } |x - x_0| < a, |y - y_0| < b \text{ for some } a, b > 0\}. \tag{7}$$

Suppose both the function $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous in R.

Under these conditions, the *Cauchy theorem* states that, for some open interval I containing x_0, there exists a unique solution $y(x)$ of ODE (6) defined on the interval I and such that $y(x_0) = y_0$.

If the regularity conditions are not satisfied, uniqueness is not guaranteed. For example:

GAP 12

Exercise 2. Determine whether the Cauchy theorem guarantees existence and uniqueness of the solution on a small interval for the following initial value problems:

$$y' = 2x^2y^2, \quad y(1) = -1, \tag{8}$$

$$y' = x \ln y, \quad y(1) = 1, \tag{9}$$

$$y' = \sqrt[3]{y}, \quad y(0) = 1. \tag{10}$$

If we know the general solution of an ODE, the Cauchy problem is solved by imposing the initial conditions and finding the corresponding value of the constant C. For example to solve the Cauchy problem

GAP 13

Exercise 3. Solve the initial value problems:

$$y' = y^2, \quad y(1) = 2, \tag{11}$$

$$y' = 2x + 3, \quad y(1) = 2. \tag{12}$$

First order ODEs can also more generally be given in *implicit form* as

$$F(x, y, y') = 0. \tag{13}$$

For example:

GAP 14

Now we want to consider some methods for finding explicit solutions.

2.2.2 Separable first order differential equations

Example 4. Consider the differential equation

$$y' = xe^{-y}. \tag{14}$$

One can check that the general solution is given by

$$y = \log\left(\frac{1}{2}x^2 + c\right). \tag{15}$$

Indeed:

GAP 15

How to find this general solution ? We can rewrite the equation in *separated variables* form:

GAP 16

A first order ODE in one variable (6) is *separable* or has its *variables separated* if can be written in the form

$$\frac{dy}{dx} = \frac{A(x)}{B(y)} \tag{16}$$

or equivalently

$$A(x)dx = B(y)dy. \tag{17}$$

The general solution is then obtained by integration:

GAP 17

Example 5. The general solution of

$$y' = \frac{x^2 + 2}{y} \tag{18}$$

is obtained by separating variables:

GAP 18

Exercise 6. Solve the initial value problems:

$$y' = -6xy, \quad y(0) = 7, \tag{19}$$

$$y' = \frac{4 - 2x}{3y^2 - 5}, \quad y(1) = 3. \tag{20}$$

2.2.3 Exact first order differential equations

Example 7. Consider now the differential equation

$$y' = \frac{x + \sin y}{2y - x \cos y}. \tag{21}$$

We can check that the solution in implicit form is given by

$$\frac{1}{2}x^2 + x \sin y - y^2 = c \tag{22}$$

for any constant c. Indeed:

GAP 19

Such type of implicit solutions can be found when the equation is *exact*, i.e. can be written in the form

$$A(x, y)dx + B(x, y)dy = 0 \tag{23}$$

with functions A and B that satisfy the exactness condition

GAP 20 .

In such case there exists a function $g(x, y)$ such that

GAP 21 .

The general solution of the equation is then given in implicit form by

$$g(x, y) = c \tag{24}$$

for any constant c.

Example 8. In the example above A and B are given by

GAP 22

which satisfy the exactness condition since

GAP 23

Let's find the function g in this case

GAP 24

Hence we obtain the implicit solution given above.

Exercise 9. Find the general solution of

$$(4y + 3x^2 - 3xy^2)y' = -6xy + y^3. \tag{25}$$

Exercise 10. Show that the exactness condition implies the existence of the function g. Hint: Integrate the first equation for g, obtaining g up to an arbitrary function $f(y)$; then write the second equation for g; show that it determines f only when the exactness condition holds.

Exercise 11. Show that (24) gives a solution to (23).

In some cases equations of the form (23) which are not exact can be transformed into an exact equation by a choice of an *integrating factor* i.e. a function $I(x, y)$ such that

$$I(x, y)[A(x, y)dx + B(x, y)dy] = 0 \tag{26}$$

is an exact equation. This equivalent form of the equation can be solved implicitly as above. In general, integrating factors are difficult to uncover.

Exercise 12. Show that the equation

$$(y^2 - y)dx + xdy = 0 \tag{27}$$

is not exact. Show that $I = 1/y^2$ is an integrating factor.

The method of integrating factor is important in the solution of *linear* first order differential equations.

2.2.4 Linear first order differential equations

A first order ODE is called *linear* when $f(x, y)$ is linear in y, i.e. has the form

$$y' + p(x)y = q(x). \tag{28}$$

This can be rewritten as

GAP 25

This is not exact, because

GAP 26

However it becomes exact by multiplying by the integrating factor

$$I(x) = e^{\int p(x)dx} \tag{29}$$

as one can check:

GAP 27

Then, to find the general solution of a first order linear ODE one first computes the integrating factor $I(x)$ and then, following the method described before for exact ODEs, obtains a function g s.t.

GAP 28

The general solution is given implicitly by

$$g(x, y) = c \tag{30}$$

for a constant c.

Example 13. Solve the ODE

$$y' + y = e^{-x} \tag{31}$$

GAP 29

Exercise 14. Solve the initial value problem

$$y' - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1. \tag{32}$$

Exercise 15. Find the general solution of

$$(x^2 + 1)y' + 3xy = 6x. \tag{33}$$

One can also obtain the following formula for the general solution of (28)

$$y = e^{-\int p(x)dx} \int q(x)e^{\int p(x)dx}dx. \tag{34}$$

Exercise 16. By computing explicitly g prove the general formula (34).

2.3 Second order differential equations

A second order ODE in implicit form

$$G(x, y, y', y'') = 0 \tag{35}$$

or in standard form:

$$y'' = f(x, y, y'). \tag{36}$$

For theoretical purposes it is useful to remark right away that a second-order ODE (36) can be regarded as a special case of a *system* of *two* first-order ODE's by introducing a new variable $v = y'$:

$$\begin{aligned} y' &= v && (37) \\ v' &= f(x, y, v) && (38) \end{aligned}$$

A general system of n first-order ODE's for n unknown functions $y_1, \ldots, y_n$ can be written in vector form as

$$\vec{y}' = \vec{F}(x, \vec{y}), \tag{39}$$

where $\vec{y} = (y_1, \ldots, y_n)$, and $\vec{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a function. The Cauchy problem in this case requires specifying the initial values of all components of $\vec{y}$, $\vec{y}(x_0) = \vec{y}_0$, and the existence and uniqueness theorem is exactly analogous to the single ODE case, with y replaced by $\vec{y}$. In particular, the for a second-order ODE one needs to fix not only the value y_0 of the function $y(x)$ at the point x_0 but also the value y_0' of the derivative $y'(x)$ at x_0:

GAP 30

In some cases the system (37)-(38) *decouples*, and we are reduced to solving two separate first order ODEs, for instance:

- If the equation does not depend on y directly but only through its derivatives, i.e. $y'' = f(x, y')$, so:

GAP 31

Example 17. Solve $y'' = x(y')^2$ for $y(0) = 1$, $y'(0) = -2$.

GAP 32

- If the equation does not depend on x directly, i.e. $y'' = f(y, y')$. In this case one introduces a new variable $v = y'$ and regards v as a function of y rather than of x. Notice that one has:

GAP 33

and so we can rewrite

GAP 34

Example 18. Solve $y'' = y^{-1}(y')^2$.

GAP 35

2.3.1 Linear second order ODEs

A linear second order ODE (i.e. G is linear in y, y', y'') can be written as

GAP 36

where the coefficient functions $A(x)$, $B(x)$, $C(x)$ and $F(x)$ are defined and continuous on some (possibly unbounded) interval I. If $A(x) \neq 0$ for each $x \in I$, we can divide it out and put the equation in the form

GAP 37

Example 19.

GAP 38

2.3.2 Existence and uniqueness

The *existence and uniqueness theorem* for second order *linear* ODEs states that: given functions $p(x)$, $q(x)$ and $f(x)$ continuous on some open interval I containing the point x_0, then, for any two numbers y_0 and y_0', there is a unique solution $y(x)$ to the initial value problem

$$\begin{cases} y'' + p(x)y' + q(x)y = f(x), \\ y(x_0) = y_0, \\ y'(x_0) = y_0'. \end{cases} \tag{40}$$

The solution is defined on the *whole* interval I.

2.3.3 Principle of superposition

If the function $f(x)$ vanishes we say the equation is *homogeneous*, otherwise it is *inhomogeneous*.

For a homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{41}$$

we have the important *principle of superposition*: if y_1 and y_2 are solutions of the homogeneous equation (41), then $c_1y_1 + c_2y_2$ is also a solution for any two constants c_1 and c_2.

Let's prove it:

GAP 39

Example 20. Check that the equation

$$y'' - 2y' + y = 0 \tag{42}$$

has $y_1 = e^x$ and $y_2 = xe^x$ as solutions. Solve the i.v.p. $y(0) = 3$, $y'(0) = 1$.

GAP 40

2.3.4 Fundamental set of solutions and the Wronskian

Suppose we have two solutions $y_1(x)$, $y_2(x)$ of the homogeneous equation (41). Then by the principle of superposition $y = c_1y_1 + c_2y_2$ is a solution, for any two constants c_1, c_2. Suppose we want to find the solution corresponding to the initial values

$$y(x_0) = y_0, \quad y'(x_0) = y_0'. \tag{43}$$

GAP 41

Given two functions f and g, the *Wronskian* of f and g is the determinant

GAP 42

Two functions y_1, y_2 defined on an open interval I are *linearly independent* if

$$c_1y_1 + c_2y_2 \equiv 0 \tag{44}$$

implies that both constants c_1 and c_2 are zero. Two linearly independent solutions form a *fundamental set of solutions*.

We have the following result (which generalizes to higher order equations): if y_1 and y_2 are two solutions to the homogeneous equation (41) on an interval I and W is their Wronskian, then:

(a) if y_1, y_2 are linearly dependent, then $W \equiv 0$ on I,

(b) if y_1, y_2 are linearly independent, then $W(x) \neq 0$ for any $x \in I$.

Let us prove statement (a):

GAP 43

and statement (b):

GAP 44

Note that a fundamental set of solutions always exists:

GAP 45

Finally we can state the theorem about the *general solution of the homogeneous equation* (41): let y_1 and y_2 form a fundamental set of solutions (i.e. they are linearly independent); then for any solution y of (41) there exist constants c_1 and c_2 s.t.

$$y = c_1 y_1 + c_2 y_2. \tag{45}$$

Proof:

GAP 46

The Wronskian W of two solutions of a homogeneous linear second order ODE (41) satisfies the equation

$$W' = -pW. \tag{46}$$

Proof:

GAP 47

Exercise 21. Use this fact to prove *Abel's formula*:

$$W(x) = W(x_0) e^{-\int_{x_0}^{x} p(\xi) d\xi}. \tag{47}$$

2.3.5 Reduction of order

If we know a solution y_1 of a second order linear ODE, we can find a second one by looking for a solution y_2 of the form $y_2 = \nu y_1$. Substituting in the original equation we can find an equation for ν which reduces to a first order ODE.

Example 22. Knowing that $y_1 = t^{-1}$ is a solution of

$$2t^2 y'' + ty' - 3y = 0 \tag{48}$$

find another solution y_2.

GAP 48

Exercise 23. Knowing that $y_1 = t$ is a solution of

$$t^2 y'' + 2ty' - 2y = 0 \tag{49}$$

find another solution y_2.

2.3.6 Inhomogeneous equations

Let y_p be a solution of the inhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{50}$$

for coefficients continuous on an interval I. Then any solution y of such inhomogeneous equation is given by

$$y = y_h + y_p, \tag{51}$$

where y_h is a solution of the associated homogeneous equation.

Proof:

GAP 49

Example 24. Find the solution of the equation

$$y'' + 4y = 12x \tag{52}$$

such that $y(0) = 5$ and $y'(0) = 7$.

GAP 50

2.3.7 Linear second order ODE with constant coefficients

Let's consider the homogeneous linear second order ODE

$$ay'' + by' + cy = 0 \tag{53}$$

where a, b and c are (real) constants.

Let's try under what conditions $e^{\lambda x}$ is a solution of such equation

GAP 51

The associated *characteristic equation* is

GAP 52

The form of the general solution depends on sign of the discriminant $\Delta = b^2 - 4ac$.

- If $\Delta > 0$ the characteristic polynomial has *distinct real* roots λ_1, λ_2. In this case the general solution is

  GAP 53

as we can easily see by computing the Wronskian

GAP 54

Example 25. Solve $2y'' - 7y' + 3y = 0$.

GAP 55

Exercise 26. Solve $y'' + 2y' = 0$.

- If $\Delta = 0$ there are *two equal real roots* λ. In this case the general solution is

GAP 56

Example 27. Solve $y'' + 2y' + y = 0$ for $y(0) = 5$, $y'(0) = -3$.

GAP 57

Exercise 28. Solve $y'' - 4y' + 4y = 0$.

- If $\Delta < 0$ then there are *two complex-conjugate roots* $\lambda_1 = \lambda$, $\lambda_2 = \bar{\lambda}$. The general real solution is obtained by imposing the reality condition $y = \bar{y}$ on
$$y = k_1 e^{\lambda x} + k_2 e^{\bar{\lambda} x}. \tag{54}$$

GAP 58

We get the general solution

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx) \tag{55}$$

where $\lambda = a + ib$.

Example 29. Solve $y'' - 4y' + 5y = 0$ for $y(0) = 1$, $y'(0) = 5$.

GAP 59

Exercise 30. Solve $y'' - 6y' + 16y = 0$.

Exercise 31. Solve the *Euler's equation*

$$Ax^2 u''(x) + Bxu'(x) + cu(x) = 0. \tag{56}$$

Hint: change the independent variable to $t = \ln x$.

2.3.8 Methods for finding particular solutions to inhomogeneous equations

To find the general solution of a inhomogeneous linear ODE

$$y'' + py' + qy = f \tag{57}$$

is sufficient to know the general solution of the associated homogeneous equation and a particular solution of the inhomogeneous equation.

There are two methods to find a particular solution of an inhomogeneous ODE.

Method of undetermined coefficients

One guesses a particular solution of the inhomogeneous ODE, starting from the form of f, leaving the coefficients undetermined. Then substitutes in the ODE to find the coefficients.

Example 32. Find a particular solution of $y'' - 4y' - 12y = 3e^{5t}$.

GAP 60

Method of variation of parameters

Let y_1 and y_2 be a set of fundamental solutions for the homogeneous ODE. Then one looks for a particular solution of the inhomogeneous ODE of the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{58}$$

Assuming

$$u_1'y_1 + u_2'y_2 = 0 \tag{59}$$

one finds that u_1 and u_2 must satisfy

$$u_1'y_1' + u_2'y_2' = f. \tag{60}$$

The last two equations can be solved for u_1 and u_2 obtaining

$$u_1 = -\int \frac{y_2 f}{W} dx, \quad u_2 = \int \frac{y_1 f}{W} dx \tag{61}$$

where W is the Wronskian of y_1, y_2. If one can perform these integrals, one thus obtains a particular solution by formula (58).

Example 33. Find a particular solutions of $y'' + 9y = 3\tan 3x$.

GAP 61

Exercise 34. Prove (60).

Exercise 35. Prove (61).

3 Partial differential equations - introduction

3.1 Basic concepts

Reading: Section 1.1 of [S]. Review A.1, A.2, A.3. Homework: all exercises from Section 1.1.

Outline: definition of PDE, order of a PDE, basic examples, linear operators, homogeneous and inhomogeneous linear equations, general solution to a PDE depends on arbitrary functions. Review of some material from calculus and linear algebra.

3.2 Linear first order PDEs

Reading: Section 1.2 of [S]. Homework: all exercises from Section 1.2.

Outline: the constant coefficient equation (geometric and coordinate method), the variable coefficient equation, characteristic curves.

3.2.1 Derivation of simple transport equation

Reading: Example 1 of Section 1.3 of [S].

4 Wave equation

4.1 The vibrating string

Reading: Example 2 of Section 1.3 of [S]. Homework: exercise 1 of Section 1.3.

4.2 Initial value problem on the line

Reading: Section 2.1 of [S]. Homework: example 2 of Section 2.1 (sketch the solution at different times t); exercises 1, 2, 5, 7.

Outline: general solution, characteristic coordinates, initial value problem, d'Alembert formula.

Exercise 1. Rewrite the wave equation in the coordinates

$$\xi = x + ct, \quad \eta = x - ct \tag{1}$$

and find its general solution. Show by switching back to the original variables x, t that the general solution is

$$u(x,t) = f(x+ct) + g(x-ct) \tag{2}$$

for arbitrary functions f and g.

Exercise 2. Prove that the *plane wave*

$$u(x,t) = Ae^{i(kx+\omega t)} \tag{3}$$

satisfies the wave equation if and only if the real parameters k (wave number) and ω (frequency) satisfy the dispersion relation

$$\omega = \pm ck. \tag{4}$$

Show that $u(x,t)$ is periodic both in x, with period $2\pi/k$, and in t, with period $2\pi/\omega$.

4.3 Causality, energy and well-posedness

Reading: Sections 2.2, 1.4, 1.5 of [S]. Homework: exercises 1, 2, 3 of Section 2.2.

Outline: dependence and influence domains, causality, Huygens principle, propagation of singularities, energy conservation, uniqueness from energy conservation, well-posed problems, stability of initial value problem.

4.4 Wave equation on the half-line

Optional reading: Section 3.2 (pages 61-62) of [S]. Homework: exercises below.

Outline: Dirichlet and Neumann boundary conditions, odd and even function, wave equation on the half-line with Dirichlet and Neumann boundary conditions, reflection of waves.

Exercise 3. Fill in the following gaps:

Consider the initial value problem for the wave equation on the half-line with Dirichlet boundary condition at $x = 0$

GAP 1

This problem can be reduced to the problem on the line by using *odd functions.* A function $f(x)$ on the real line is *odd* if

GAP 2

i.e. if its graph is symmetric w.r.t. the origin. If f is an odd function, then $f(0) = 0$.

First, extend the functions ϕ, ψ to odd functions ϕ_{odd}, ψ_{odd} on the real line

GAP 3

Second, solve the i.v.p. on the line by D'Alembert formula

GAP 4

The solution $v(x, t)$ is odd, indeed:

GAP 5

hence satisfies the boundary condition for all t.

Finally, the solution to the original b.v.p. is given by restriction of the solution $v(x,t)$ to the half-line.

We can now write the solution in terms of the initial data $\phi(x)$, $\psi(x)$ (rather than in terms of their odd extensions). For $t > 0$, $x > ct$ the solution is

GAP 6

For $t > 0$, $0 < x < ct$ the solution is

GAP 7

Exercise 4. Sketch the domain of dependence.

Exercise 5. Sketch the behaviour of a triangular initial profile (pinched string) in the case of Dirichlet boundary condition at $x = 0$.

Exercise 6. Solve the i.v.p. on the half-line with Neumann boundary conditions at $x = 0$

$$u_x(0,t) = 0, \quad \text{for all } t \tag{5}$$

using *even* functions. Sketch the dependence and influence domains. Sketch the behaviour of a triangular initial profile.

Exercise 7. (Ex. 5 p. 66 of [S]) Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) \equiv 1$, $u_t(x,0) \equiv 0$. Find the location of the singularity of the solution.

4.5 Wave equation on an interval and separation of variables

Reading: Sections 4.1, 4.2 of [S], skipping the parts on the diffusion equation that will be considered later. Homework: exercises 4, 5 of Section 4.1, exercise 2 of Section 4.2.

Outline: separated solutions, Dirichlet and Neumann boundary conditions on the interval, eigenvalue problems, sine and cosine Fourier series.

4.6 Periodic problem for the wave equation

A function $f(x)$ is periodic of period L if $f(x+L) = f(x)$ for any x. Consider the initial value problem for the wave equation on the line with periodic initial data, i.e. both $\phi(x)$ and $\psi(x)$ are periodic functions of period $2l$.

Exercise 8. Show that the solution $u(x,t)$ is periodic of period $2l$ in x, for any t. (Use uniqueness of the i.v.p. for the wave equation on the line. Alternatively use the d'Alembert formula).

Let us begin by looking for separated solutions, i.e. of the form

$$u(x,t) = X(x)T(t) \tag{6}$$

where $X(x)$ is a periodic function of period $2l$. As before we get the following equations for X and T

$$-X'' = \lambda X, \quad T'' + \lambda c^2 T = 0. \tag{7}$$

Exercise 9. Solve the eigenvalue problem for X, finding all the eigenvalues λ_n and the corresponding eigenfunctions X_n.

One finds that the eigenvalues are

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, \quad n = 0, 1, 2, \ldots \tag{8}$$

To the eigenvalue $\lambda_0 = 0$ corresponds the eigenfunction $X_0 = 1$, and to each eigenvalue λ_n, $n \geqslant 1$ correspond two linearly independent eigenfunctions

$$\cos\frac{\pi n x}{l}, \qquad \sin\frac{\pi n x}{l}. \tag{9}$$

Exercise 10. For each eigenvalue, find the corresponding general solution of the equation for T. Write down the separated solutions u_n to the wave equation with periodic boundary conditions.

Combining the separated solutions we obtain the following general solution

$$u(x,t) = \frac{A_0}{2} + \frac{C_0}{2}t + \sum_{n\geqslant 1} \cos\frac{\pi n c}{l}t\left(A_n\cos\frac{\pi n}{l}x + B_n\sin\frac{\pi n}{l}x\right) \tag{10}$$

$$+ \sum_{n\geqslant 1} \sin\frac{\pi n c}{l}t\left(C_n\cos\frac{\pi n}{l}x + D_n\sin\frac{\pi n}{l}x\right). \tag{11}$$

The initial conditions $\phi(x)$ and $\psi(x)$ are expressed as full Fourier series.

5 Fourier series

5.1 Classical Fourier series

Reading: Sections 5.1 and 5.2 of [S]. Examples 1 - 6 and exercises 2, 3, 9 from Section 5.1; exercises 2, 3, 4, 5, 6, 11, 17 from Section 5.2.

Outline: sine, cosine and full Fourier series, orthogonality and coefficients formulas, solution of the wave equation initial value problems with Dirichlet, Neumann and periodic boundary conditions. Complex form of the Fourier series.

5.2 Orthogonality and general Fourier series

Reading: Sections 5.3 of [S]. Exercises 2, 3, 5, 9, 10 from Section 5.3.

Outline: Green's second identity, symmetric boundary conditions, complex eigenvalues, negative eigenvalues.

5.3 Convergence theorems

Reading: Sections 5.4, 5.5 (p.136-9) of [S].

Outline: pointwise, uniform and mean square convergence, convergence theorems for classical and general Fourier series, least-square approximation, Bessel's inequality, Parseval's equality, proof of pointwise convergence.

5.4 Incomplete notes

5.4.1 Different types of convergence for series of functions

Definition of pointwise, uniform and L^2 convergence of a series of functions over an interval $[a, b]$. See [S] p.126.

5.4.2 Convergence theorems

- pointwise convergence
  - standard convergence theorem for complex F. s. [D]

5.4.3 Some L^2 theory

Inner product and norm on L^2. Consider two $\mathbb{C}$-valued functions on the interval (a, b). The *Hermitian inner product* is

GAP 1

The L^2-*norm* is

GAP 2

We use $\|f - g\|_2$ as a measure of the "distance" of two functions f, g on (a, b).

Least-square approximation theorem. Consider a sequence X_n, $n = 1, 2, \ldots$ of orthogonal functions i.e.

GAP 3

Let $f(x)$ be a function on (a, b) with $\|f\| < \infty$ and $N > 0$ a fixed integer. Recall that the Fourier coefficients A_n of $f(x)$ w.r.t. the functions X_n are defined by

GAP 4

The least-square approximation theorem states that

Theorem. *For any choice of complex numbers* $c_1, \dots, c_N$ *one has*

GAP 5

This result can be interpreted as follows: the combination of $X_1, \dots, X_N$ that best approximates in L^2-sense the function $f(x)$ is that given by using the Fourier coefficients A_n.

Proof. For simplicity let us prove the theorem in the case of real valued X_n, orthogonal w.r.t. the real version of the inner product

GAP 6

and assume that the X_n are normalized of length 1

$$\|X_n\|_2 = 1. \tag{1}$$

Consider the L^2-norm squared

GAP 7

rewrite it using the inner product

GAP 8

and then, using orthogonality, observe that it can be written as

GAP 9

This quantity clearly has minimum for $c_n = A_n$. □

Bessel's inequality. If we choose $c_n = A_n$ in the previous proof, we get *Bessel's inequality*

GAP 10

Parseval's equality.

5.4.4 Proof of pointwise convergence

We follow Section 5.5 of [S].

Let f be a continuous function on $\mathbb{R}$ of period 2π. The (classical) full Fourier series in this case is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx\right), \tag{2}$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny \; dy, \quad n \geqslant 0, \tag{3}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny \; dy, \quad n \geqslant 1. \tag{4}$$

We want to prove that

Theorem. *Let $f \in C^1(\mathbb{R})$ of period 2π. Then the Fourier series (2) converges pointwise to $f(x)$.*

Proof. Let S_N be the Nth partial sum

$$S_N = \frac{1}{2}A_0 + \sum_{n=1}^{N} \left(A_n \cos nx + B_n \sin nx\right). \tag{5}$$

Let us first rewrite $S_N(x) - f(x)$ in a more convenient form.

Lemma. *We have*

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\sin((N+\frac{1}{2})\theta) \; d\theta \tag{6}$$

for

$$g(\theta) = \frac{f(x+\theta) - f(x)}{\sin(\theta/2)}. \tag{7}$$

This lemma is proved by simple manipulations, using only the assumption of periodicity of $f(x)$, see p.137-9 of [S].

We need to prove that for any fixed x, the integral in (6) tends to zero as $N \to \infty$.

First observe that

Lemma. *The functions*

$$\phi_N(\theta) = \sin((N + \frac{1}{2})\theta). \tag{8}$$

are orthogonal with respect to the inner product

$$(f, g) = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\ d\theta. \tag{9}$$

Then Bessel's inequality implies

$$\frac{1}{\pi}\sum_{N=1}^{\infty} |(g, \phi_N)|^2 \leqslant \|g\|_2^2, \tag{10}$$

where we have used the fact that $\|\phi_N\|_2^2 = \pi$. So, if $\|g\|_2$ is finite, the series on the left-hand side converges, and we have that

$$|S_N(x) - f(x)|^2 = \frac{1}{4\pi}|(g, \phi_N)|^2 \to 0 \tag{11}$$

for $N \to \infty$. It remains to check that g has finite L^2 norm. But since

$$\lim_{\theta \to 0} g(\theta) = 2f'(\theta), \tag{12}$$

$g(\theta)$ is a continuous function on $(-\pi, \pi)$, and the integral of $|g|^2$ is finite, i.e. $\|g\|_2 < \infty$. $\square$

6 Heat equation

6.1 Physical derivation of the heat equation

Reading: See Example 4 of Section 1.3 of [S] for the derivation of the diffusion equation, Example 5 for the heat equation.

Let $u(x,t)$ be the temperature at the point x and at time t of a rod, assumed insulated except at the ends. The amount of heat ΔH in the small interval $(y, y + \Delta y)$ is proportional to the temperature

$$\Delta H = c\rho u(y,t)\Delta y$$

where c is the *specific heat* and ρ is the linear mass density. The heat in the interval (x_0, x) is obtained by integration:

$$H = \int_{x_0}^{x} c\rho u(y,t)\ dy.$$

Fourier's law states that the heat flows from hot to cold regions with the rate proportional to the gradient of the temperature. By our assumption this can only occur at the ends, so we have:

$$\frac{dH}{dt} = \int_{x_0}^{x} c\rho u_t(y,t)\ dy = \kappa u_x(x) - \kappa u_x(x_0),$$

where κ is a proportionality constant (the *heat conductivity*). By differentiating w.r.t. x we obtain the *heat equation*:

$$u_t = k u_{xx}, \tag{1}$$

for an unknown function $u(x,t)$, where $k = \frac{\kappa}{c\rho}$.

Exercise 1. To which physical configurations correspond the boundary conditions of Dirichlet or Neumann type ?

6.2 Maximum principle, uniqueness and stability

Reading: section 2.3 of [S]. Exercises 1, 2, 4, 6.

Outline: maximum and minimum (weak and strong) principles, proof of weak maximum principle, well-posedness of initial value problem on the interval with Dirichlet b.c., uniqueness and stability in the uniform sense.

6.2.1 Strong and weak maximum and minimum principles

Consider a solution $u(x,t)$ to the heat equation

$$u_t = k u_{xx} \tag{2}$$

on a rectangle R given by $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$ and denote by R' the initial line $t = 0$ and the sides of the rectangle $x = 0$, $x = l$.

The *strong maximum principle* states that, if $u(x,t)$ is not constant, the maximum M of $u(x,t)$ is assumed only on R' (and in particular $u(x,t) < M$ for all $(x,t) \in R \setminus R'$).

The *weak maximum principle* states that the maximum M of $u(x,t)$ is assumed on R' (hence $u(x,t) \leqslant M$ for $(x,t) \in R$).

The strong and weak *minimum principles* are completely analogous.

Exercise 2. State the strong and weak minimum principles.

Exercise 3. Prove the minimum principle for a solution $u(x,t)$ of the heat equation on a rectangle R by applying the maximum principle to $-u(x,t)$.

6.2.2 Proof of the weak maximum principle

We only give the proof of the weak maximum principle. See section 2.3 of [S].

Let $u(x,t)$ be a solution of the heat equation on the rectangle R given by $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$ and denote by R' the initial line $t=0$ and the sides of the rectangle $x=0$, $x=l$.

Let M be the maximum of $u(x,t)$ on R'.

Let us consider a *perturbation* $v(x,t)$ of the solution $u(x,t)$ by adding to it ϵx^2, where $\epsilon > 0$ is a small positive constant

$$v(x,t) = u(x,t) + \epsilon x^2. \tag{3}$$

A simple computation shows that $v(x,t)$ satisfies the *diffusion inequality* at all points in R

$$v_t - k v_{xx} = -2k\epsilon < 0. \tag{4}$$

Let us now show that $v(x,t)$ *cannot* have a maximum in $R \setminus R'$:

- Suppose $v(x,t)$ has a maximum at the interior point (x_0, t_0). Then $v_t = 0$ and $v_{xx} \leqslant 0$ at (x_0, t_0), and this contradicts the diffusion inequality.
- Suppose $v(x,t)$ has a maximum at the top edge point (x_0, T). At such point the t derivative has to be nonnegative, because

$$v_t(x_0, T) = \lim_{\delta \to 0^-} \frac{v(x_0, T) - v(x_0, T-\delta)}{\delta} \geqslant 0. \tag{5}$$

  Then $v_t \geqslant 0$ and $v_{xx} \leqslant 0$ at (x_0, T), and this contradicts the diffusion inequality.

Since $v(x,t)$ must have a maximum in R, but it cannot be in $R \setminus R'$, then it can only be on R'.

On R' we have

$$v(x,t) \leqslant u(x,t) + \epsilon l^2 \leqslant M + \epsilon l^2, \tag{6}$$

therefore

$$v(x,t) \leqslant M + \epsilon l^2 \tag{7}$$

on all R.

By definition of $v(x,t)$ we have that

$$u(x,t) \leqslant M + \epsilon(l^2 - x^2) \tag{8}$$

on all R, but this implies $u(x,t) \leqslant M$. □

6.2.3 Uniqueness and stability

Exercise 4. Prove the uniqueness of the initial value problem for the heat equation on an interval with Dirichlet boundary conditions using the maximum and minimum principles.

Exercise 5. Prove the stability in uniform sense of the initial value problem for the heat equation on an interval with Dirichlet boundary conditions using the maximum and minimum principles.

6.3 Energy method

Reading: sections 1.5, 2.3 of [S].

Outline: energy method, uniqueness and stability in the mean-square sense.

Exercise 6. Prove the uniqueness of the initial value problem for the heat equation on an interval with Dirichlet boundary conditions using the energy method.

Exercise 7. Prove the stability in mean-square sense of the initial value problem for the heat equation on an interval with Dirichlet boundary conditions using the energy method.

6.4 Heat equation on the whole line: Green's function

Reading: section 2.4 of [S]. Exercises 1, 2, 3, 4, 5, 9, 10, 11, 13, 15.

Exercise 8. Prove that the *Green's function*

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}} : \tag{9}$$

1. satisfies the heat equation for $t > 0$;
2. is even in x;
3. has the limit
$$\lim_{t \to 0^+} S(x,t) = \begin{cases} 0 & x \neq 0, \\ +\infty & x = 0. \end{cases} \tag{10}$$

6.5 The heat equation on the half-line

Reading: section 3.1 of [S]. Exercises: 1, 2, 3.

6.6 Separation of variables for the heat equation on the interval

Reading: sections 4.1, 4.2 of [S]. Exercises 2, 3 of section 4.1, exercises 1, 3, 4 of section 4.2.

Exercise 9. Solve the initial value problem for the following equation on the interval $0 < x < \pi$ for $t > 0$:

$$\begin{cases} u_t = u_{xx} - 4u, \\ u_x(0,t) = u(\pi,t) = 0, \\ u(x,0) = x^2 - \pi^2. \end{cases}$$

6.7 Smoothing property of the heat equation

Reading: Section 3.5 of [S].

We state a rigorous result about the solution formula (**??**):

Theorem. *Let $\phi(x)$ be a bounded continuous function on the real line. The solution formula* (**??**) *defines an infinitely differentiable function $u(x,t)$ for $t > 0$ and $x \in \mathbb{R}$, which satisfies the heat equation* (1)*. Moreover*

$$\lim_{t\to 0^+} u(x,t) = \phi(x).$$

Notice that, in sharp contrast with the case of the wave equation, a continuous but singular initial data becomes smooth as soon as $t > 0$. This is known as the *smoothing property* of the heat equation.

The theorem above holds true even for piecewise continuous initial data, but in that case the limit for $t \to 0^+$ gives

$$\lim_{t\to 0^+} u(x,t) = \frac{1}{2}(\phi(x^+) + \phi(x^-)).$$

For a proof see section 3.5 of [S].

6.8 Solutions to some exercises

6.8.1 [S] §2.3 Ex. 1

Can easily check that $u(x,t) = 1 - x^2 - 2kt$ solves $u_t = ku_{xx}$ by taking derivatives. The locations of the maximum and minimums on the rectangle $(x,t) \in R = [0,1] \times [0,T]$ can be found as usual by evaluating derivatives. $u_t = -2k$ is never vanishing, so critical points cannot be in the interior of R, confirming the maximum principle. On the line $t = 0$ we have $u = 1 - x^2$ which has maximum 1 at $(0,0)$. Similarly for $x = 0$ we have $u = 1 - 2kt$ and for $x = 1$ we get $u = -2kt$. So the maximum is at $(0,0)$.

6.8.2 [S] §2.3 Ex. 2

(a) We know the maximum $M(T)$ is located either on the initial line for $0 \leqslant x \leqslant l$ or on the sides of the rectangle $x = 0, l$ for $0 \leqslant t \leqslant T$. If we increase T, i.e. we consider a new rectangle with $T' > T$ either we get a new maximum on the longer sides or we still have the same maximum as before. Hence $M(T)$ can only increase (or stay constant) as a function of T. (b) The same reasoning implies that the minimum $m(T)$ can only decrease or stay constant as a function of T.

6.8.3 [S] §2.3 Ex. 4

(a) Use the maximum and minimum principles. (b) Define $w(x,t) = u(1-x,t)$ and show that it satisfies the same equation and the same initial and boundary conditions as $u(x,t)$. Hence by uniqueness they must coincide. (c) Compute the derivative of the energy and show that it is equal to a strictly negative quantity.

6.8.4 [S] §2.3 Ex. 6

Let $w = u - v$. Notice that w is positive for $t = 0$, $x = 0$ and $x = l$, hence its minimum must also be positive. Finally use the minimum principle.

6.8.5 [S] §2.4 Ex. 1

The solution formula gives

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} \, dy. \tag{11}$$

We want the exponent to be in the standard form $-p^2$, hence we perform the change of variable

$$p = \frac{x-y}{\sqrt{4kt}}. \tag{12}$$

The integral becomes

$$\sqrt{4kt} \int_{\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2} \, dp. \tag{13}$$

Splitting the integral in two parts and using the definition of error function we get

$$u = \frac{1}{2} \operatorname{Erf} \frac{x+l}{\sqrt{4kt}} - \frac{1}{2} \operatorname{Erf} \frac{x-l}{\sqrt{4kt}}. \tag{14}$$

6.8.6 [S] §2.4 Ex. 2

By manipulation of the integral we obtain

$$u = 2 - \frac{1}{2} \operatorname{Erf} \frac{-x}{\sqrt{4kt}} - \frac{3}{2} \operatorname{Erf} \frac{x}{\sqrt{4kt}}. \tag{15}$$

6.8.7 [S] §2.4 Ex. 3

By completing the square as in Example 2 of [S] we get

$$u = e^{3x+9kt}. \tag{16}$$

6.8.8 [S] §2.4 Ex. 4

$$u = \frac{1}{2} e^{-x+kt} \left(1 - \operatorname{Erf} \frac{-x+2kt}{\sqrt{4kt}} \right). \tag{17}$$

6.8.9 [S] §2.4 Ex. 9

Following the steps given in the exercise we obtain

$$u = x^2 + 2kt. \tag{18}$$

6.8.10 [S] §2.4 Ex. 10

As explained in the text, by a change of variable we obtain

$$u = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2 dp \tag{19}$$

and comparing with previous exercise the integral equals $\sqrt{\pi}/2$.

6.8.11 [S] §3.1 Ex. 1

As usual by odd extension we get to formula (6) on page 59 of [S]. Substituting $\phi(x) = e^{-x}$ and expressing as usual the integrals in terms of the error function we get

$$u = \frac{1}{2} e^{-x+kt} \left(1 - \operatorname{Erf} \frac{-x+2kt}{\sqrt{4kt}} \right) - \frac{1}{2} e^{x+kt} \left(1 - \operatorname{Erf} \frac{x+2kt}{\sqrt{4kt}} \right). \tag{20}$$

6.8.12 [S] §3.1 Ex. 2

Consider the constant solution $u(x,t) \equiv 1$ on the whole line and the odd solution of Example 1 of page 59 of [S]. Combine them linearly to obtain a solution that satisfies the correct initial and boundary conditions on the half-line.

Answer: $u = 1 - \operatorname{Erf} \frac{x}{\sqrt{4kt}}$.

7 Fourier transform

7.1 Heuristic derivation from Fourier series

Reading: Section 12.3 p.343-344 [S] for the details.

We begin by deriving the inversion formula for the Fourier transform from the Fourier series of a periodic function of period $2l$ by taking the limit $l \to \infty$.

Following the steps outlined in the textbook and during lecture, we obtain the following inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-iky} dy \right) e^{ikx} dx. \tag{1}$$

7.2 Definition of Fourier and inverse Fourier transforms

Let $f(x)$ be a complex valued function on the real line. We define the *Fourier transform* of $f(x)$ as a complex valued function on the real line given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{2}$$

Conversely, given a function $g(k)$ on the real line the *inverse Fourier transform* of $g(k)$ is defined as

$$\check{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{ikx} dk. \tag{3}$$

Exercise 1. Prove that the Fourier transform (and the inverse Fourier transform) define *linear* maps.

The inversion formula (1) says that if $\hat{f}(k)$ is the Fourier transform of a function $f(x)$, then the inverse Fourier transform of $\hat{f}(k)$ gives back the original function $f(x)$. Below we will reformulate this statement in a more precise way. Let us first go through some examples of Fourier transforms.

Remark. Here we follow the convention of placing the factor of $(2\pi)^{-1}$ in front of the integral for the inverse Fourier transform. Different text use different conventions in this regard.

7.3 Examples of Fourier transforms

Exercise 2. Compute the Fourier transform of the rectangular pulse[1]

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a. \end{cases} \tag{4}$$

Example 3. The Fourier transform of the Gaussian function

$$f(x) = e^{-\frac{x^2}{2}}. \tag{5}$$

[1]Solution: $\hat{f}(p) = 2p^{-1} \sin(pa)$.

The Fourier transform is given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx. \tag{6}$$

Completing the square as follows

$$-\frac{x^2}{2} - ikx = -\frac{1}{2}(x+ik)^2 - \frac{k^2}{2}, \tag{7}$$

using the identity

$$\int_{-\infty}^{\infty} e^{-\frac{(x+ik)^2}{2}}\, dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx \tag{8}$$

and the Gaussian integral (10), we obtain

$$\hat{f}(k) = \sqrt{2\pi} e^{-\frac{k^2}{2}}. \tag{9}$$

Exercise 4. Using integration over a contour in the complex plane prove formula (8).

Exercise 5. Prove Gaussian integral (or Euler-Poisson integral) formula

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}. \tag{10}$$

Exercise 6. Compute the Fourier transform of the exponentially decaying pulse[2]

$$f_r(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases} \tag{11}$$

for $a > 0$.

Exercise 7. Compute the Fourier transform of [3]

$$f_l(x) = \begin{cases} e^{ax} & x < 0 \\ 0 & x > 0 \end{cases} \tag{12}$$

for $a > 0$.

Exercise 8. Combining $f_r(x)$ and $f_l(x)$ from the previous two exercises and using linearity of Fourier transform show that

$$\mathcal{F}[e^{-a|x|}] = \frac{2a}{k^2 + a^2}. \tag{13}$$

Exercise 9. Show that

$$\mathcal{F}[\text{sign}(x) e^{-a|x|}] = \frac{-2ik}{k^2 + a^2}. \tag{14}$$

Exercise 10. By taking the $a \to 0^+$ limit in the previous exercise show that

$$\mathcal{F}[\text{sign}(x)] = -\frac{2i}{k}. \tag{15}$$

[2] Solution: $\hat{f}_r(k) = (ik + a)^{-1}$.

[3] Solution: $\hat{f}_l(k) = (-ik + a)^{-1}$.

Exercise 11. Compute the Fourier transform of

$$f(x) = \begin{cases} \frac{1}{2a}\cos\omega x & |x| < a, \\ 0 & |x| > a \end{cases} \tag{16}$$

where a is a positive constant.

7.4 Properties of the Fourier transform

A few important properties of the Fourier transform are given in the following exercises.

Let $f(x)$ be a function of the real line and $\hat{f}(k)$ its Fourier transform.

Exercise 12. Show that the Fourier transform of the shifted function $f(x-\xi)$ for $\xi \in \mathbb{R}$ is

$$e^{-ik\xi}\hat{f}(k). \tag{17}$$

Exercise 13. Show that the Fourier transform of $e^{i\xi x}f(x)$ for $\xi \in \mathbb{R}$ is

$$\hat{f}(k-\xi). \tag{18}$$

Exercise 14. Show that the Fourier transform of $f(cx)$ for $c > 0$ is

$$\frac{1}{c}\hat{f}(\frac{k}{c}). \tag{19}$$

Exercise 15. Using the previous result compute the Fourier transform of

$$e^{-\frac{ax^2}{2}}. \tag{20}$$

A particularly important property is the fact that *the Fourier transform maps the derivative to the multiplication by ik*. More precisely let us derive w.r.t x the inverse Fourier transform formula[4]

GAP 1

We see that

$$\mathcal{F}[f'(x)] = ik\hat{f}(k). \tag{21}$$

A similar calculation, deriving the Fourier transform formula w.r.t p, gives *the Fourier transform of the product $xf(x)$*:

GAP 2

The *convolution* of two functions $f(x)$, $g(x)$ on the real line is a function $(f * g)(x)$ on $\mathbb{R}$ defined by

[4]We assume the integrand is nice enough to bring the derivative inside the integral.

GAP 3

Exercise 16. Prove that the Fourier trasform $\mathcal{F}[f * g]$ of the convolution of two functions $f(x)$, $g(x)$ is given by the product $\hat{f}(k)\hat{g}(k)$ of their Fourier transforms.

Exercise 17. Prove the following properties of the convolution

$$f * g = g * f \tag{22}$$

$$(f * g)' = f' * g = f * g' \tag{23}$$

$$(f * g) * h = f * (g * h) \tag{24}$$

Exercise 18. Derive the Fourier transform (9) of the Gaussian as follows: first, write a first order ordinary differential equation for $f(x) = e^{-\frac{x^2}{2}}$; second, take the Fourier transform and solve the resulting ODE in the variable k; finally fix the integration constant using (10).

7.5 Derivation of Poisson formula using the Fourier transform

Consider the initial value problem for the heat equation on the real line

GAP 4

Let us consider the Fourier transform of the heat equation and of the initial condition (w.r.t. the variable x)

GAP 5

The resulting problem is a first order ODE for $\hat{u}(p, t)$ that can be easily solved

GAP 6

Performing the inverse Fourier transform we can express the solution as a convolution

$$u(x, t) = \phi * \psi \tag{25}$$

Finally we obtain the inverse Fourier transform of $\hat{\psi}(p,t) = e^{-ktp^2}$

GAP 7

The resulting formula is the Poisson integral

GAP 8

Exercise 19. Consider the ODE for a function $u(x)$ on the line $-\infty < x < \infty$

$$-u_{xx} + \omega^2 u = h(x) \tag{26}$$

with a given forcing function $h(x)$. By taking the Fourier transform of the ODE find a solution $u(x)$ in terms of the Fourier transform $\hat{h}(k)$ of $h(x)$.

7.6 Inversion theorem for Fourier transform

The Fourier transform can be applied to different types of functions for which the definition (2) makes sense.

We consider here the simplest case where $f(x)$ is a complex valued integrable continuous function and also the Fourier transform is an integrable function. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is called (absolutely) integrable if

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty. \tag{27}$$

Theorem. *If $f(x)$ is a continuous integrable function and its Fourier transform $\hat{f}(k)$ is also an integrable function, then the inversion formula (1) holds true, i.e.,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk = f(x). \tag{28}$$

This is the simplest and most important version of the inversion theorem, and the one we have to keep in mind. Let's see what we can say in some other case.

- Suppose we relax the requirement that $\hat{f}(k)$ is also integrable, but we ask that $f(x)$ is differentiable. In such case we have:

  Theorem. *If $f(x)$ is a differentiable integrable function, then*

  $$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \hat{f}(k)e^{ikx}dk = f(x). \tag{29}$$

- Suppose now we allow both $f(x)$ and $f'(x)$ to be piecewise continuous. In such case we have:

Theorem. *If $f(x)$ is an integrable function, and both $f(x)$ and the derivative $f'(x)$ are piecewise continuous, then*

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(k) e^{ikx} dk = \frac{1}{2}(f(x^+) + f(x^-)). \tag{30}$$

So, in particular, if x is a point where $f(x)$ is continuous, we have the same result as above. It at x there is a jump the inverse Fourier transform converges to the average of the left and right limits of f at x.

- Another important case is when f is a square integrable function, see later for some results in this case.

Exercise 20. Consider again Exercise 2. Applying the inversion theorem for a piecewise continuous function with piecewise continuous derivative, show that the inverse Fourier transform of $\hat{f}(p)$ has value $1/2$ at $x = \pm a$.

Exercise 21. Using the inversion theorem and the result of Exercise 8, evaluate the following nontrivial integral

$$\int_{-\infty}^{\infty} \frac{e^{ipx}}{p^2 + a^2}\, dp. \tag{31}$$

7.7 Fourier transform for square integrable functions

We say that a complex valued function $f(x)$ on $\mathbb{R}$ is *square integrable* if

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty. \tag{32}$$

The functions $f : \mathbb{R} \to \mathbb{C}$ which are square integrable form a vector space which is denoted $L^2(\mathbb{R})$. Notice that a square integrable function does not have to be differentiable or even continuous.

Exercise 22. Show that the functions $f(x) = 1$ and $f(x) = e^{ikx}$, $k \in \mathbb{R}$ are not in $L^2(\mathbb{R})$.

Exercise 23. Show that any piecewise continuous $\mathbb{C}$-valued function $f(x)$ on $\mathbb{R}$ such that $|f(x)| \leqslant M|x|^{-1/2-\delta}$ for large $|x|$, for some $M > 0$, $\delta > 0$, belongs to $L^2(\mathbb{R})$.

Given two complex valued functions f, g on $\mathbb{R}$ we define their *Hermitian inner product* as

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx. \tag{33}$$

The L^2*-norm* of a complex valued function on $\mathbb{R}$ is defined as

$$\|f\|_2 = (f, f)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{34}$$

Clearly a function f is square integrable if and only if $\|f\|_2 < \infty$.

On the space of square integrable functions $L^2(\mathbb{R})$ two important inequalities hold true: the Cauchy-Schwarz and the triangle inequality.

Theorem. *The Hermitian inner product and the norm on $L^2(\mathbb{R})$ satisfy the Cauchy-Schwarz inequality*

$$|(f, g)| \leqslant \|f\|_2 \|g\|_2 \tag{35}$$

and the triangle inequality

$$\|f + g\|_2 \leqslant \|f\|_2 + \|g\|_2 \tag{36}$$

for every $f, g \in L^2(\mathbb{R})$.

Exercise 24. Prove Cauchy-Schwarz inequality.

Exercise 25. Using Cauchy-Schwarz inequality prove the triangle inequality.

Remark. The Cauchy-Schwarz inequality and the triangle inequality are the most important inequalities in functional analysis, and in particular they are true not only in the case of $L^2(\mathbb{R})$ but for any complex vector space with an Hermitian inner product.

The Fourier transform is well-behaved on the the space of square integrable functions. We have in particular:

Theorem. *The Fourier transform of a square integrable function $f(x)$ is a well-defined square integrable function $\hat{f}(k)$.*

We have therefore that the Fourier transform defines a linear map of $L^2(\mathbb{R})$ to itself:

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{37}$$

Moreover the Fourier transform preserves the Hermitian inner product in $L^2(\mathbb{R})$. This is the statement of the *Parseval's equality*:

Theorem. *If $f, g \in L^2(\mathbb{R})$, denote $\hat{f}$, $\hat{g}$ their Fourier transforms, then*

$$(f, g) = \frac{1}{2\pi}(\hat{f}, \hat{g}) \tag{38}$$

or, more explicitly,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}\, dk. \tag{39}$$

We say that the Fourier transform is a *unitary* map from $L^2(\mathbb{R})$ to itself.

Exercise 26. Show that a function $f \in L^2(\mathbb{R})$ and its Fourier transform $\hat{f}(k)$ have the same L^2-norm, up to a factor $(2\pi)^{-\frac{1}{2}}$.

Exercise 27. Show that if two square integrable functions are orthogonal, then also their Fourier transforms are orthogonal.

Remark. The coefficient $(2\pi)^{-1}$ is due to our choice of constant in front of the Fourier and inverse Fourier transforms.

Remark. Of course, as expected, the inverse Fourier transform also maps square integrable functions to square integrable functions, and preserves the inner product. Moreover, morally it is indeed the inverse of the map (37). However, to be really precise in this case we should take into account the fact that a function f in $L^2(\mathbb{R})$ can have zero norm, i.e., $\|f\|_2 = 0$. If we identify two functions in $L^2(\mathbb{R})$ if their difference has zero norm, then the statement above is rigorously true, i.e., the inverse Fourier transform give the inverse of the the map (37).

7.8 Solutions to some exercises

7.8.1 Ex. 11

Let

$$\cos \omega x = \frac{1}{2}(e^{i\omega x} + e^{-i\omega x}) \tag{40}$$

in the definition of Fourier transform and get

$$\hat{f}(k) = \frac{1}{4a}\int_{-a}^{a} (e^{i(w-k)x} + e^{-i(w+k)x})\ dx. \tag{41}$$

By integrating the exponentials this is equal to

$$\frac{1}{2a}\frac{\sin(\omega-k)a}{w-k} + \frac{1}{2a}\frac{\sin(\omega+k)a}{w+k}. \tag{42}$$

7.8.2 Ex. 19

Taking the Fourier transform of the ODE we get

$$p^2\hat{u} + \omega^2\hat{u} = \hat{h} \tag{43}$$

that we can easily solve

$$\hat{u}(p) = \frac{\hat{h}(p)}{p^2+\omega^2}. \tag{44}$$

Recalling that $\mathcal{F}[e^{-\omega|x|}] = \frac{2\omega}{p^2+\omega^2}$, and that the Fourier transform of the convolution of two functions is given by the product of the Fourier transforms, we get

$$u = \mathcal{F}^{-1}[\hat{h}\cdot\frac{1}{p^2+\omega^2}] = h(x) * \mathcal{F}^{-1}[\frac{1}{p^2+\omega^2}] = \frac{1}{2\omega}h(x) * e^{-\omega|x|}. \tag{45}$$

7.8.3 [S] §12.3 Ex.6

1. By definition of inverse Fourier transform and using the fact that $f(x)$ is band-limited we have

   $$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{ikx}\ dk. \tag{46}$$

   Evaluating this at $x = -n$ we get exactly the definition of the Fourier coefficient c_n, for which we know

   $$\hat{f}(k) = \sum_{n\in\mathbb{Z}} c_n e^{ink} = \sum_{n\in\mathbb{Z}} f(n)e^{-ink}. \tag{47}$$

   Substituting this in (46), exchanging the sum and the integral, and integrating the exponential we get the desired result.

2. A simple computation of the inverse Fourier transform of $\hat{f}(k)$ gives

   $$f(x) = \frac{\sin \pi x}{\pi x}. \tag{48}$$

For $n \in \mathbb{Z}$, $n \neq 0$ the sine vanishes so

$$f(n) = 0, \quad n \neq 0, \tag{49}$$

but for $n = 0$ we get an indeterminate form, that is evaluated using (for example) l'Hopital's rule

$$f(0) = \lim_{x\to 0} \frac{\sin \pi x}{\pi x} = \frac{0}{0} = \lim_{x\to 0} \frac{\pi \cos \pi x}{\pi} = 1. \tag{50}$$

7.8.4 [S] §12.3 Ex.7

1. Since $f(x)$ vanishes for large x, the sum in $\sum_n f(x + 2\pi n)$ is finite for every x, hence the function $g(x)$ is well-defined. Then

$$g(x + 2\pi) = \sum_{n\in\mathbb{Z}} f(x + 2\pi n + 2\pi) = \sum_{n\in\mathbb{Z}} f(x + 2\pi n) = g(x), \tag{51}$$

   hence $g(x)$ is periodic.

2. By definition of Fourier coefficients of a 2π-periodic function

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-imx}\, dx \tag{52}$$

$$= \frac{1}{2\pi} \sum_n \int_{-\pi}^{\pi} f(x + 2\pi n) e^{-imx}\, dx \tag{53}$$

   that by changing variable of integration to $y = x + 2\pi n$ is equal to

$$\frac{1}{2\pi} \sum_n \int_{-\pi+2\pi n}^{\pi+2\pi n} f(y) e^{-imy}\, dy. \tag{54}$$

   This is clearly equal to $F(m)/2\pi$, where $F(k)$ is the Fourier transform of $f(x)$.

3. The Fourier series for $g(x)$ is then

$$g(x) = \frac{1}{2\pi} \sum_n F(n) e^{inx}. \tag{55}$$

   Setting $x = 0$ we get the desired result.

8 Laplace equation

8.1 Laplace and Poisson equations

Reading: Section 6.1 of [S]. Exercises 1, 5, 6.

Let us consider the Laplace equation in two dimensions. Although it differs from the wave equation only by a sign, the properties of this PDE are completely different.

The Laplacian operator Δ in this case is

GAP 1

The Laplace equation is

GAP 2

A real-valued function $u(x, y)$ that satisfies the Laplace equation is called *harmonic*.

The inhomogeneous version of the Laplace equation is called *Poisson equation*

GAP 3

Remark. The Laplace equation in one dimension is trivial:

GAP 4

Notice that there is no preferred "time" or "space" variable in the case of Laplace equation. A consequence of this is that there is no well-posed initial value problem. The correct problems associated to the Laplace equation are boundary value problems of Dirichlet and Neumann type.

Dirichlet boundary value problem:

GAP 5

Exercise 1. State the Dirichlet problem in one dimension and find its solution.

Example 2. The wave equation in $2+1$ dimensions (2 spatial and 1 time variables) is

$$u_{tt} = u_{xx} + u_{yy}. \tag{1}$$

The *stationary* solutions are those that do not depend on time t. It follows that the stationary solutions of the wave equation are harmonic functions in two variables.

Example 3. Let $f = u + iv : \mathbb{C} \to \mathbb{C}$ be a complex valued function on the complex plane. It is *analytic* if it is expressible as power series in $z = x + iy \in \mathbb{C}$

$$f(z) = \sum_{n \geqslant 0} a_n z^n. \tag{2}$$

Deriving with respect to y, resp. x, we get

$$\frac{\partial f}{\partial y} = i \sum_{n \geqslant 0} n a_n z^{n-1} = i \frac{\partial f}{\partial x}. \tag{3}$$

It follows that u, v satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x. \tag{4}$$

Therefore both u and v are harmonic functions, i.e.,

$$u_{xx} = v_{yx} = -u_{yy}, \tag{5}$$

and similarly for v.

8.2 The maximum principle

Let D be a connected bounded open subset of $\mathbb{R}^2$, let $\bar{D} = D \cup \partial D$ be its closure and ∂D its boundary. Let u be a harmonic function on D which is continuous on $\bar{D}$.

The *maximum principle* states that, if u is not constant, then the maximum of u is achieved only on the boundary ∂D.

More explicitly, there exists $x_M \in \partial D$ s.t. $u(x) \leqslant u(x_M)$ for all $x \in \bar{D}$ and $u(x) < u(x_M)$ for all $x \in D$.

The proof of the maximum principle will be given later using the mean value property.

Exercise 4. State the analogous *minimum principle*. Prove it using the maximum principle.

Exercise 5. Let $B(x,r)$ be the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^2$

$$B(x,r) = \{y \in \mathbb{R}^2 \mid |x - y| < r\}. \tag{6}$$

Recall the definitions of open, closed, bounded and connected subset of $\mathbb{R}^2$. Write the definition of closure $\bar{D}$ and boundary ∂D of an open set $D \subset \mathbb{R}^2$.

Exercise 6. Prove uniqueness of the solution of the Dirichlet problem for the Laplace equation using the maximum principle.

Exercise 7. Prove the continuous dependence on the boundary data (i.e. stability) of the Dirichlet problem for the Laplace equation: let u_1, u_2 be two harmonic functions on a domain D as above. Suppose that

$$|u_1(x) - u_2(x)| \leqslant \epsilon \text{ for } x \in \partial D. \tag{7}$$

Using maximum (and minimum) principle, show that

$$|u_1(x) - u_2(x)| \leqslant \epsilon \text{ for } x \in \bar{D}. \tag{8}$$

Exercise 8. The *weak* maximum principle states that the maximum of a harmonic function has to be achieved on the boundary ∂D. Prove the weak maximum principle by deforming the harmonic function u (as in the case of the weak maximum principle for the heat equation), i.e., let

$$v(x, y) = u(x, y) + \epsilon(x^2 + y^2) \tag{9}$$

for $\epsilon > 0$. Deduce that v cannot have a maximum in D. Finally conclude that u has maximum on ∂D. See p. 155 of [S].

8.3 Separation of variables on a rectangle

Reading: Section 6.2 of [S]. Example 1, exercises: 1, 3, 4, 5.

Consider the Laplace equation $u_{xx} + u_{yy} = 0$ in the rectangle $R = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } 0 < x < a, 0 < y < b\}$.

Let us look for separated solutions $u(x, y) = X(x)Y(y)$. Substituting we get two ODEs

GAP 6

Depending on the sign of the separating constant λ, we obtain different types of solutions

GAP 7

Note that using the superposition principle for linear equations we can impose the inhomogeneous b.c. one at the time. First impose the homogeneous b.c. on three sides of the rectangle, and find the separated solutions. By combining them find the general solution, then impose the inhomogeneous b.c.

Exercise 9. Solve the Dirichlet problem for the Laplace equation on the rectangle R with the b.c.

$$u(x,0) = f(x), \quad u(x,b) = u(0,y) = u(a,y) = 0. \tag{10}$$

Hint: impose first the homogeneous (i.e. vanishing) b.c., then obtain a general solution in the form of a series, and finally impose the remaining b.c.

Exercise 10. Find the solution $u(x,y)$ to the Dirichlet b.v.p. for the Laplace equation in the rectangle

$$0 < x < a, \quad 0 < y < b \tag{11}$$

with b.c.

$$u(0,y) = y(b-y), \quad u(x,0) = \sin\frac{\pi x}{a}, \quad u(x,b) = u(a,y) = 0. \tag{12}$$

Hint: write u as the sum of two solutions u_1 and u_2, each having an inhomogeneous b.c. only on one side of the rectangle.

8.4 Separation of variables on a circle and Poisson's formula

Reading: Section 6.3 of [S]. Exercise 1, 2.

Exercise 11. Show that the Laplacian operator is invariant under *translations*

$$x' = x + a, \quad y' = y + b, \tag{13}$$

and *rotations by an angle* α

$$\begin{cases} x' = x\cos\alpha + y\sin\alpha, \\ y' = -x\sin\alpha + y\cos\alpha \end{cases} \tag{14}$$

in the plane. *(See p.156 in [S].)*

Exercise 12. The polar coordinates (r,θ) on $\mathbb{R}^2$ are defined by

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases} \tag{15}$$

Show that the Laplacian Δ in polar coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{16}$$

(See p.156-7 of [S])

Exercise 13. Show that the harmonic functions that are rotationally invariant, i.e., that in polar coordinates do not depend on θ, are given by

$$u(r,\theta) = c_1 \log r + c_2. \tag{17}$$

Exercise 14. Find the separated solutions to the Laplace equation on a disk in polar coordinates.

Exercise 15. Find the general solution by taking an infinite linear combination of the separated solutions with arbitrary coefficients. Find the coefficients in terms of the boundary value $h(\theta)$ on the circle, using periodic Fourier series formulas.

Exercise 16. Show that the infinite sum obtained in the previous exercise can be summed.

By performing separation of variables in polar coordinates we have obtained the *Poisson's formula*

GAP 8

One can prove that, given a continuous function h on the circle of radius a, the Poisson's formula defines a function u which is harmonic in the interior of the disc of radius a, and continuous on the closure of the disk, and the limit of $u(x)$ for x approaching the boundary point y is given by $h(y)$.

Exercise 17. By using separation of variables in polar coordinates (r, θ), find the harmonic function on the annular domain

$$a < r < b \tag{18}$$

with the boundary conditions

$$u|_{r=a} = 1, \qquad \left(\frac{\partial u}{\partial r}\right)_{r=b} = (\cos\theta)^2. \tag{19}$$

8.5 Mean value property

Let u be a harmonic function on an open subset D of $\mathbb{R}^2$. Denote $C_{x,a}$ (resp. $D_{x,a}$) the circle (resp. the disk) of center x and radius a

GAP 9

If $C_{x,a}$ is contained in D, then the mean of u on $C_{x,a}$ is defined as the integral

GAP 10

Let $D_{x,a} \subset D$; the *mean value property* states that the mean of u on the circle $C_{x,a}$ centered at x is equal to the value of u at the center x.

Let (r, θ) be polar coordinates centered at the point x. The proof is simply obtained by setting $r = 0$ in the Poisson formula with boundary condition $h(\theta) = u(a, \theta)$.

Exercise 18. Let the notations be as above. Argue why the value of u inside $D_{x,a}$ is given by the Poisson formula with $h(\theta) = u(a, \theta)$.

Exercise 19. Show that a (trivial) version of the mean value property holds for the one-dimensional Laplace equation.

Exercise 20. Using the mean value property prove the strong maximum principle. See. p. 169 of [S].

8.6 Some solutions to exercises

8.6.1 [S] §6.1 Ex. 5

$$u = \frac{r^2}{4} - \frac{a^2}{4} \tag{20}$$

8.6.2 Ex. 9

$$u(x, y) = \sum_{n>0} \frac{B_n}{\sinh\left(\frac{\pi n}{a} b\right)} \sinh\left(\frac{\pi n}{a}(b - y)\right) \sin\left(\frac{\pi n}{a} x\right) \tag{21}$$

$$B_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{\pi n}{a} x\right) dx \tag{22}$$

8.6.3 Ex. 10

$$u(x, y) = \sum_{n>0} A_n \sinh\left(\frac{\pi n}{b}(x - a)\right) \sin\left(\frac{\pi n}{b} y\right) + \frac{\sinh\left(\frac{\pi}{a}(y - b)\right)}{\sinh\left(-\frac{\pi b}{a}\right)} \sin\left(\frac{\pi x}{a}\right) \tag{23}$$

$$A_n = \frac{\frac{4b^2}{(n\pi)^3}(1 - (-1)^n)}{\sinh\left(-\frac{\pi n a}{b}\right)} \tag{24}$$

8.6.4 Ex. 17

By writing the Laplace equation in polar coordinates (r, θ) and looking for separated solutions $u(r, \theta) = R(r)\Theta(\theta)$ we obtain two ODEs

$$-\Theta'' = \lambda\Theta, \quad r^2 R'' + rR' = \lambda R. \tag{25}$$

Since Θ has to be 2π-periodic, the first ODE is an eigenvalue problem, with eigenvalues $\lambda_n = n^2$ for $n \geqslant 0$ and corresponding eigenfunctions

$$\Theta_n^{(1)} = \cos n\theta, \quad \Theta_n^{(2)} = \sin n\theta \tag{26}$$

for $n > 0$ and

$$\Theta_0 = 1 \tag{27}$$

for $n = 0$.

The general solution to the second ODE corresponding to the eigenvalue λ_n for $n > 0$ is obtained by substituting $R = r^\alpha$ and solving the characteristic equation $\alpha^2 = n^2$, which implies $\alpha = \pm n$, so there are two linearly independent solutions

$$r^n, \quad r^{-n}. \tag{28}$$

The case $n = 0$ is solved by separation of variables. It has two linearly independent solutions

$$1, \quad \log r. \tag{29}$$

Since our original problem has two inhomogeneous boundary conditions, we must split it as sum of two problems with only one inhomogemeous b.c.

Problem 1: Consider the boundary conditions

$$u|_{r=a} = 0, \quad \frac{\partial u}{\partial r}|_{r=b} = \cos^2\theta. \tag{30}$$

The homogeneous b.c. implies that

$$R(a) = 0. \tag{31}$$

Imposing this b.c. on the general solution of the second ODE found above we get

$$R_n(r) = \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n, \text{ for } n > 0, \tag{32}$$

$$R_0(r) = \log\frac{r}{a}, \text{ for } n = 0. \tag{33}$$

Combining the separated solutions in an infinite sum (we can do this because we have only imposed homogeneous b.c. so far), we get the general solution of Problem 1:

$$u(r,\theta) = \frac{A_0}{2}\log\frac{r}{a} + \sum_{n>0}\left(\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right)(A_n\cos n\theta + B_n\sin n\theta). \tag{34}$$

Now we can find the coefficients A_n, B_n in terms of the inhomogeneous b.c. Using the standard formulas for the full Fourier series we get

$$A_0 = \frac{b}{\pi}\int_0^{2\pi}\cos^2\theta\ d\theta, \tag{35}$$

$$\frac{n}{a}A_n\left(\left(\frac{b}{a}\right)^{n-1} + \left(\frac{b}{a}\right)^{-n-1}\right) = \frac{1}{\pi}\int_0^{2\pi}\cos n\theta\cos^2\theta\ d\theta \tag{36}$$

for $n > 0$, and $B_n = 0$.

By using the trigonometric identity $2\cos^2\theta = 1 + \cos 2\theta$, we get that

$$A_0 = b, \quad A_2 = \frac{a}{4}\frac{1}{\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^{-3}} \tag{37}$$

and the remaining A_n vanish.

Hence the solution of Problem 1 is

$$u_1 = \frac{b}{2}\log\frac{r}{a} + \frac{a}{4}\frac{\left(\frac{r}{a}\right)^2 - \left(\frac{a}{r}\right)^2}{\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^{-3}}\cos 2\theta. \tag{38}$$

Problem 2: We now consider the boundary conditions

$$u|_{r=a} = 1, \quad \frac{\partial u}{\partial r}|_{r=b} = 0. \tag{39}$$

This problem can be solved as above, or by simply noticing that $u \equiv 1$ is a solution that satisfies the b.c.

The solution of the whole exercise is

$$u = 1 + u_1. \tag{40}$$

9 Laplace transform

Reading: Section 12.5 of [S]. Examples 2, 3 p.354. Exercises 3, 6, 7 of Section 12.5.

Given a function $f(t)$, its Laplace transform $F(s)$, also denoted $\mathcal{L}[f(t)]$ is

$$F(s) = \int_0^\infty f(t)e^{-st}\ dt. \tag{1}$$

Example 1. Compute the Laplace transform of $f(t) = 1$. By definition

$$F(s) = \int_0^\infty e^{-st}\ dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s} \tag{2}$$

which converges for $s > 0$.

Exercise 2. Compute the Laplace transform of $f(t) = e^{at}$.

Exercise 3. Compute the Laplace trasform of $f(t) = \sin at$ and $f(t) = \cos at$.

Exercise 4. Compute the Laplace trasform of $f(t) = \sinh at$ and $f(t) = \cosh at$.

Exercise 5. Compute the Laplace transform of $H(t - b)$, where

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases} \tag{3}$$

Exercise 6. Show that the Laplace transform is a linear operator.

Example 7. The Laplace transform of the derivative of $f(t)$ is

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0). \tag{4}$$

To show this we use integration by parts

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Exercise 8. Using formula (4) show that

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \tag{5}$$

Exercise 9. Show that the Laplace transform of $e^{bt}f(t)$ is $F(s - b)$.

Exercise 10. By deriving the definition of Laplace transform w.r.t. s, show that

$$\mathcal{L}[tf(t)] = -F'(s). \tag{6}$$

Exercise 11. Using previous exercise show that

$$\mathcal{L}[t^k] = (-\partial_s)^k \frac{1}{s}. \tag{7}$$

Derive that

$$\mathcal{L}[t^k] = \frac{k!}{s^{k+1}}. \tag{8}$$

Exercise 12. By a change of integration variable show that for $c > 0$

$$\mathcal{L}[f(ct)] = \frac{1}{c}F(\frac{s}{c}). \tag{9}$$

Exercise 13. Compute $\mathcal{L}[tf'(t)]$.

Exercise 14. Show that the Laplace transform of the "convolution" of two functions $g(t)$ and $f(t)$ i.e.

$$\int_0^t g(t-t')f(t')\ dt' \tag{10}$$

is given by the product of their Laplace transforms $G(s)F(s)$. Hint: Write definition of Laplace transform of (10) and change the order of integration in the resulting double integral, paying attention to the extremes of integration.

The Laplace transform can be used to reduce initial value problems for ODEs to algebraic problems. Let us consider a simple example to illustrate the method.

Example 15. Solve the initial value problem

$$y'' - 5y' + 6y = 0, \quad y(0) = y'(0) = 2. \tag{11}$$

Denote $Y(s)$ the Laplace transform of $y(t)$. Applying the Laplace transform to the ODE above and taking into account the initial conditions we obtain

$$Y(s) = \frac{2s-8}{s^2-5s+6}. \tag{12}$$

Rewriting the result a sum of simple fractions and recalling the form of the Laplace transform of the exponential we get

$$Y(s) = \mathcal{L}[4e^{2t} - 2e^{3t}]. \tag{13}$$

Hence the solution to the original i.v.p. is

$$y(t) = 4e^{2t} - 2e^{3t}. \tag{14}$$

Exercise 16. Solve the following i.v.p.

$$y'' + 3y' + 2y = H(x), \quad y(0) = y'(0) = 0. \tag{15}$$

Exercise 17. Solve the following i.v.p. for a non-constant coefficients ODE[5]

$$ty'' - ty' + y = 2, \quad y(0) = 2, y'(0) = -4. \tag{16}$$

[5]Solution: $y = 2 - 4t$.

10 Green's functions

Reading: [S], Sections 7.2 (esp. representation formulas (1) and (5)), 7.3 (esp. Theorems 1 and 2), 7.4. Exercises 1,3 of Section 7.2, Exercises 1,2,5,6,7,8,11 of Section 7.4.

11 Classification of second order linear PDEs

11.1 Comparison wave, heat and Laplace equations

See section 2.5 of [S].

11.2 General second order linear PDEs in two variables

See section 1.6 of [S].

A Tests 2013

A.1 First test

Partial Differential Equations - AUC
First exam - 1/10/2013

1. Find the general solution of
$$\frac{dy}{dx} - \frac{3}{x^2}y = \frac{1}{x^2} \tag{1}$$
by finding the integrating factor $I(x,y)$ and solving the resulting exact equation.

2. Let $\mathcal{L}$ be a linear operator. Let u_1, u_2 be solutions of the linear homogeneous equation $\mathcal{L}(u) = 0$ and v a solution of the linear inhomogeneous equation
$$\mathcal{L}(u) = g. \tag{2}$$
Show that for any constants c_1, c_2 the function $c_1u_1 + c_2u_2 + v$ is a solution of the inhomogeneous equation (2).

3. Consider the PDE
$$u_x + x^2 u_y = 0 \tag{3}$$
for the unknown function $u(x,y)$.
   (a) Find the characteristic curves in the xy-plane.
   (b) Write down the general solution in terms of an arbitrary function of one variable f.
   (c) Check directly that such solution satisfies the equation (1).
   (d) Find the solution $u(x,y)$ that satisfies the auxiliary condition
   $$u(x,0) = \arctan x. \tag{4}$$
   (e) Find the general solution of the inhomogeneous equation
   $$u_x + x^2 u_y = y + x^3. \tag{5}$$
   (Hint: look for a particular solution of the form $u = x^n y^m$ for some n, m)

4. Write down the simple transport equation describing the concentration $w(x,t)$ of a pollutant in a pipe containing a liquid moving to the left with constant speed v. If at time $t = 0$ the concentration is given by $w(x,0) = e^{-x^2}$, what is the concentration at time $t = T$?

5. Find the solution of the wave equation $u_{tt} = 2u_{xx}$ with initial conditions
   (a) $u(x,0) = e^{-x^2}, \quad u_t(x,0) = 0$;
   (b) $u(x,0) = 0, \quad u_t(x,0) = \frac{1}{\pi(4+x^2)}$.

A.2 Midterm test

Partial Differential Equations - AUC

Midterm exam - 22/10/2013

1. Consider the initial value problem for the wave equation on the half-line $0 < x < \infty$ with Neumann boundary conditions at $x = 0$:

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < \infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u_x(0,t) = 0 \end{cases} \tag{1}$$

   (a) Define even functions ϕ_e, ψ_e that extend ϕ, ψ to the real line and show that the solution $u(x,t)$ given by d'Alembert formula with initial data ϕ_e, ψ_e is also even.
   (b) Show that such solution, restricted on the half line $x > 0$, solves the i.v.p. (1).
   (c) Write the formula for $u(x,t)$ in terms of ϕ, ψ in the cases $x > ct$ and $x < ct$ (assume $t > 0$).
   (d) Sketch in (x,t)-plane the dependence domain for a point (x_0, t_0) with $t_0 > 0$, $x_0 > ct_0$.
   (e) Sketch in (x,t)-plane the dependence domain for a point (x_0, t_0) with $t_0 > 0$, $0 < x_0 < ct_0$.
   (f) Sketch the influence domain of an interval $[a,b]$ for $0 < a < b$.
   (g) Repeat the last three questions in the case of Dirichlet b.c. at $x = 0$.

2. Prove energy conservation for a solution $u(x,t)$ of the wave equation on the half-line (describing a string of linear mass density ρ and tension T) with Dirichlet boundary condition $u(0,t) = 0$, and with initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \qquad x > 0 \tag{2}$$

   which vanish for $x > R > 0$.

3. Find:

   (a) The Fourier sine series of $\phi(x) = \frac{\pi}{4} - \frac{x}{2}$ on the interval $(0, \pi)$.
   (b) The Fourier cosine series of $\phi(x) = \frac{\pi}{4} - \frac{x}{2}$ on the interval $(0, \pi)$.
   (c) The full Fourier series of the periodic function ϕ of period 2π defined by

$$\phi(x) = \begin{cases} x & 0 < x < \pi \\ \pi & \pi < x < 2\pi \end{cases} \tag{3}$$

   and extended periodically.

4. Consider the following equation on the interval $[0, l]$, with boundary conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} + ru \quad r < 0 \\ u(0,t) = 0 \\ u_x(l,t) = 0. \end{cases} \tag{4}$$

   (a) Find the separated solutions $u_n = X_n(x)T_n(t)$. In particular consider the associated eigenvalue problem, find all eigenvalues λ_n and eigenfunctions X_n, and solve the associated equation for T_n.

   (b) Write the infinite series solution of (4) and find the series representation of the initial conditions $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$ at $t = 0$.

   (c) State the definition of symmetric boundary conditions for the eigenvalue problem $-X'' = \lambda X$ on the interval $[0, l]$. Show that the boundary conditions in (4) are symmetric.

   (d) Show that the symmetry of the boundary conditions implies orthogonality of the eigenfunctions X_n.

   (e) Use orthogonality to express the coefficients in the series expansions of the initial conditions in terms of ϕ and ψ.

A.3 Third test

Partial Differential Equations - AUC

Third exam - 19/11/2013

In this test there are 5 questions.

1. Consider the infinite series of functions over the interval $[a, b]$

$$\sum_{n=1}^{\infty} f_n(x) \tag{1}$$

   and let $f(x)$ be a function on $[a, b]$. Give the definition of pointwise and uniform convergence of the series $\sum_{n=1}^{\infty} f_n(x)$ to the function $f(x)$.

2. Let X_n, $n = 1, 2, \ldots$ be a sequence of real orthogonal functions on the interval (a, b) w.r.t. the inner product

$$(f, g) = \int_a^b f(x)g(x)\, dx. \tag{2}$$

   Assume X_n are normalized i.e. $\|X_n\|_2 = 1$, where the L^2-norm is defined by

$$\|f\|_2 = \sqrt{(f, f)}. \tag{3}$$

   State the Bessel inequality.

3. Let $u(x,t)$ be a solution to the heat equation on the rectangle R given by

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant t \leqslant 2. \tag{4}$$

Suppose that $u(x,t) \leqslant M$ for all $(x,t) \in R$, and that $u(1,1) = M$. Explain what we can say about $u(x,t)$ according to the strong maximum principle.

4. Consider the heat equation on the interval $(0,l)$ with Neumann boundary conditions. Find the separated solutions.

5. Let $u(x,t)$ be the solution of the initial value problem for the heat equation on the real line

$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty, \ t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{5}$$

Write the solution formula (Poisson integral) for $u(x,t)$. What can we say about the continuity of the third derivative $\frac{\partial^3 u}{\partial x^3}(x,t)$ for $t > 0$? Compute explicitly (in terms of the error function) the solution for the initial condition

$$\phi(x) = \begin{cases} 1 & x > 0 \\ 2 & x < 0. \end{cases} \tag{6}$$

A.4 Final test

Partial Differential Equations - AUC

Final exam - 17/12/2013

In this test there are 3 questions. Each sub-question contributes 10%.

1. (a) Compute the Fourier transform $\hat{f}(p)$ of

$$f(x) = \begin{cases} \frac{1}{2A} & |x| < A, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

According to the inversion theorem, what value assumes the inverse Fourier transform of $\hat{f}(p)$ at $x = \pm A$?

(b) Show that the Fourier transform of the convolution $f * g$ of two functions f and g is given by the product of their Fourier transforms $\hat{f}$, $\hat{g}$.

(c) Apply the Fourier transform to the initial value problem for the heat equation on the line

$$\begin{cases} u_t = u_{xx} & t > 0, \ x \in \mathbb{R}, \\ u(x,0) = \phi(x). \end{cases} \tag{2}$$

Show that the solution $u(x,t)$ is given by the convolution of $\phi(x)$ with the inverse Fourier transform $\psi(x,t)$ of the Gaussian function $\hat{\psi}(p,t) = e^{-p^2 t}$.

(d) Write the formula for the Fourier transform of the Gaussian function. Substituting in the previous formula, find the solution formula for the heat equation on the line.

2. (a) If the Laplace transform of $f(t)$ is given by $F(s)$, what is the Laplace transform of $f'(t)$? Derive it.

   (b) The Laplace transform of $\sin t$ is $\frac{1}{s^2+1}$. Use the formula above to obtain the Laplace transform of $\cos t$.

   (c) Using the Laplace transform method, solve the initial value problem

$$2\frac{dy}{dt} - y = \sin t, \quad y(0) = 0. \tag{3}$$

3. (a) Let u be a harmonic function on an open bounded connected set $D \subset \mathbb{R}^2$ and let ∂D denote the boundary of D.

   State the maximum principle for u.

   Use the maximum principle to prove uniqueness for the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ u = h & \text{on } \partial D. \end{cases} \tag{4}$$

   (b) A harmonic function u on the unit disk $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 1\}$ when restricted to the unit circle $\{x^2 + y^2 = 1\}$ is equal to $u_{|r=1} = y = \sin\theta$, where (r, θ) are the polar coordinates. Using the Poisson formula or the mean value property, find the value of u at the origin.

   (c) Find the harmonic function $u(x, y)$ on the square $D = \{0 < x < 1, 0 < y < 1\}$ such that

$$u_x = 0 \text{ for } x = 0 \text{ and } x = 1, \quad u = 0 \text{ for } y = 0 \tag{5}$$

   and

$$u = \frac{1}{2} + 2\cos(2\pi x) \text{ for } y = 1. \tag{6}$$

B Solutions Tests 2013

B.1 First test

1. The integrating factor is
$$I = e^{3/x}. \tag{1}$$
Multiplication by I gives an exact equation that we can integrate in the usual way obtaining
$$e^{3/x}(y + \frac{1}{3}) = c \tag{2}$$
or explicitly
$$y = \frac{1}{3} + ce^{-3/x}. \tag{3}$$

2. Trivial, check the book.

3. (a) Suppose $y(x)$ is the equation for a characteristic curve, then
$$0 = \frac{du(x, y(x))}{dx} = u_x + u_y y_x = u_y(y_x - x^2), \tag{4}$$
hence
$$y_x = x^2 \tag{5}$$
so that the characteristic curves have equations
$$y = \frac{x^3}{3} + c. \tag{6}$$
(b)
$$u(x, y) = f(y - \frac{x^3}{3}) \tag{7}$$
(c) Trivial.
(d) Imposing the initial condition in the general solution above
$$u(x, 0) = \arctan x = f(-\frac{x^3}{3}). \tag{8}$$
By a change of variable $z = -\frac{x^3}{3}$ we get
$$f(z) = \arctan \sqrt[3]{-3z} \tag{9}$$
and finally the solution
$$u(x, t) = \arctan \sqrt[3]{x^3 - 3y}. \tag{10}$$
(e) Following the hint, one finds that $u = xy$ is a particular solution. Then the general solution is
$$u(x, y) = f(y - \frac{x^3}{3}) + xy. \tag{11}$$

4. The transport equation is
$$w_t = vw_x. \tag{12}$$
The general solution is
$$w(x,t) = f(x+vt). \tag{13}$$
Imposing the initial condition we get
$$w(x,0) = e^{-x^2} = f(x), \tag{14}$$
and finally
$$w(x,t) = e^{-(x+vt)^2}. \tag{15}$$

5. (a) Using d'Alembert formula with $c^2 = 2$, $\psi = 0$, $\phi(x) = e^{-x^2}$ we have
$$u(x,t) = \frac{1}{2}\left(e^{-(x+\sqrt{2}t)^2} + e^{-(x-\sqrt{2}t)^2}\right). \tag{16}$$
(b) Again by d'Alembert formula
$$u(x,t) = \frac{1}{4\sqrt{2}\pi}\left(\arctan(\frac{x}{2}+\frac{t}{\sqrt{2}}) - \arctan(\frac{x}{2}-\frac{t}{\sqrt{2}})\right). \tag{17}$$

B.2 Midterm test

1. (a) We did it at lecture, just a change of variable in d'Alembert formula
$$u(x,t) = \frac{1}{2}(\phi_e(x+ct)+\phi_e(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_e(s)\ ds. \tag{18}$$
(b) The restriction of $u(x,t)$ to $x>0$ still solves the wave equation with the correct initial data. Moreover u is even in x at all t, so u_x is odd in x, hence $u_x(0,t) = 0$, and the b.c. are satisfied.
(c) For $t>0$ and $x-ct>0$, we also have $x+ct>0$, hence d'Alembert formula is simply
$$u(x,t) = \frac{1}{2}(\phi(x+ct)+\phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\ ds. \tag{19}$$
For $t>0$ and $x-ct<0$, we still have $x+ct>0$, so d'Alembert formula becomes
$$u(x,t) = \frac{1}{2}(\phi(x+ct)+\phi(ct-x)) + \frac{1}{2c}\int_{0}^{x+ct}\psi(s)\ ds + \frac{1}{2c}\int_{0}^{-x+ct}\psi(s)\ ds. \tag{20}$$
(d)
(e)
(f)
(g)

2. We did this at lecture, only difference here is that the energy is given by the integral of the energy density on the half line.

3. (a) Sine Fourier series coefficients, $n > 0$

$$A_n = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{4} - \frac{x}{2}\right) \sin nx \; dx \tag{21}$$

This integral is easily computed (using at some point integration by parts) and is equal to

$$\frac{1}{n} \text{ for } n \text{ even} \tag{22}$$

and 0 otherwise. Hence the series is

$$\phi(x) = \sum_{m \geqslant 1} \frac{1}{2m} \sin 2mx. \tag{23}$$

(b) In this case, $n \geqslant 0$

$$A_n = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{4} - \frac{x}{2}\right) \cos nx \; dx. \tag{24}$$

This gives

$$\phi(x) = \sum_{m \geqslant 0} \frac{2}{\pi(2m+1)^2} \cos(2m+1)x. \tag{25}$$

(c) The result is

$$\phi(x) = \frac{A_0}{2} + \sum_{n \geqslant 1} (A_n \cos nx + B_n \sin nx) \tag{26}$$

for

$$A_0 = \frac{3}{2}\pi, \quad A_n = \frac{1}{n^2\pi}((-1)^n - 1), \quad B_n = -\frac{1}{n}, \quad n > 0. \tag{27}$$

4. (a) The eigenvalues are

$$\lambda_n = (\frac{\pi}{2l} + \frac{\pi n}{l})^2, \quad n \geqslant 0 \tag{28}$$

and the eigenfunctions

$$X_n = \sin(\frac{\pi}{2l} + \frac{\pi n}{l})x. \tag{29}$$

The corresponding T equation is

$$T'' + (\lambda_n c^2 - r)T = 0 \tag{30}$$

which has characteristic polynomial with roots

$$\sigma = \pm i\beta_n, \quad \beta_n := \sqrt{\lambda_n c^2 - r} \tag{31}$$

with β_n real, since $\lambda_n c^2 - r > 0$. Hence

$$T_n = A_n \cos \beta_n t + B_n \sin \beta_n t \tag{32}$$

and the separated solutions are

$$u_n = (A_n \cos \beta_n t + B_n \sin \beta_n t) \sin(\frac{\pi}{2l} + \frac{\pi n}{l})x. \tag{33}$$

(b) Taking infinite linear combinations of the separated solutions

$$u = \sum_{n\geqslant 0}(A_n \cos\beta_n t + B_n \sin\beta_n t)\sin(\frac{\pi}{2l} + \frac{\pi n}{l})x, \tag{34}$$

which gives

$$\phi(x) = u(x,0) = \sum_{n\geqslant 0} A_n \sin(\frac{\pi}{2l} + \frac{\pi n}{l})x = \sum_{n\geqslant 0} A_n X_n \tag{35}$$

and

$$\psi(x) = u_t(x,0) = \sum_{n\geqslant 0} B_n\beta_n \sin(\frac{\pi}{2l} + \frac{\pi n}{l})x = \sum_{n\geqslant 0} B_n\beta_n X_n. \tag{36}$$

(c) Formula (5) on p.119 of [S] with $a = 0$, $b = l$. Clearly it is satisfied for the b.c. in this exercise.

(d) See p.118-119 of [S].

(e) Orthogonality

$$(X_n, X_m) = \delta_{n,m}(X_m, X_m) \tag{37}$$

implies

$$A_n = \frac{(X_n, \phi)}{(X_n, X_n)} \tag{38}$$

and

$$B_n = \frac{1}{\beta_n}\frac{(X_n, \psi)}{(X_n, X_n)}. \tag{39}$$

An integration by parts shows that

$$(X_n, X_n) = \int_0^l \left(\sin(\frac{\pi}{2l} + \frac{\pi n}{l})x\right)^2 \, dx = \frac{l}{2}. \tag{40}$$

B.3 Third test

1. See book p.125-6.

2.
$$\sum_{n=1}^{N} |A_n|^2 \leqslant \|f\|_2^2 \tag{41}$$

for

$$A_n = (f, X_n). \tag{42}$$

3. The maximum appears in the interior of the square R, hence u is constant.

4. Studying the eigenvalue problem $-X'' = \lambda X$ with Neumann b.c. we get the eigenvalues

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, \quad n \geqslant 0 \tag{43}$$

with eigenfunctions

$$X_n = \cos\frac{\pi n}{l}x. \tag{44}$$

Solving the equation for T we get the separated solutions

$$u_n = e^{-(\pi n/l)^2 kt} \cos \frac{\pi n}{l} x \tag{45}$$

for $n \geqslant 0$.

5. By the smoothing property of the heat equation all the derivative exist and are continuous for $t > 0$. The formula for the solution of the heat equation on the line yields, after some manipulations of the integral

$$u(x,t) = \frac{3}{2} - \frac{1}{2} \operatorname{Erf} \frac{x}{\sqrt{4kt}}. \tag{46}$$

B.4 Final test

1. (a) A simple integration of exponential function gives

$$\hat{f}(p) = \frac{1}{2A} \int_{-A}^{A} e^{-ipx} = \frac{\sin Ap}{Ap}. \tag{47}$$

By the inversion theorem for piecewise continuous functions, the inverse Fourier transform $g(x)$ at the jump points is equal to the average of the left and right limits i.e.

$$g(\pm A) = \frac{1}{4A}. \tag{48}$$

(b) See p. 347 of the book.

(c) By applying the Fourier transform to the heat equation and the initial condition we get

$$\begin{cases} \hat{u}_t = -p^2 \hat{u}, \\ \hat{u}(p,0) = \hat{\phi}(p), \end{cases} \tag{49}$$

where $\hat{u}(p,t)$ is the Fourier transform of $u(x,t)$ in x and $\hat{\phi}(p)$ is the Fourier transform of $\phi(x)$. The solution of the ODE in t is clearly

$$\hat{u}(p,t) = \hat{\phi}(p) e^{-p^2 t} = \hat{\psi}(p,t)\hat{\phi}(p) = \mathcal{F}[\psi * \phi]. \tag{50}$$

By taking the inverse Fourier transform we conclude.

(d) We know that

$$\mathcal{F}[e^{-x^2/2}] = \sqrt{2\pi} e^{-p^2/2} \tag{51}$$

and by the rescaling property of the Fourier transform we get

$$\mathcal{F}[e^{-(ax)^2/2}] = \frac{\sqrt{2\pi}}{a} e^{-p^2/(2a^2)}. \tag{52}$$

Let $a = 1/\sqrt{2t}$ and get $\psi = (4\pi t)^{-1/2} e^{-x^2/(4t)}$. By substitution get

$$u(x,t) = \psi * \phi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} \phi(y)\, dy. \tag{53}$$

2. (a) Simple derivation by parts.

(b)
$$\mathcal{L}[\cos t] = \mathcal{L}[(\sin t)'] = -(\sin t)|_{t=0} + s\mathcal{L}[\sin t] = \frac{s}{s^2+1}. \tag{54}$$

(c) Call $Y(s)$ the Laplace transform of $y(t)$. The Laplace transform of the equation is
$$2(sY(s) - y(0)) - Y(s) = \frac{1}{s^2+1}, \tag{55}$$
hence
$$Y(s) = \frac{1}{(s^2+1)(2s-1)}. \tag{56}$$
Rewriting this as a simple fraction
$$Y(s) = \frac{4}{5}\frac{1}{2s-1} - \frac{2}{5}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1}. \tag{57}$$
The inverse Laplace transform yields
$$y = \frac{2}{5}e^{t/2} - \frac{2}{5}\cos t - \frac{1}{5}\sin t. \tag{58}$$

3. (a) See the book.

(b) By the mean value property we have just to compute the average
$$\frac{1}{2\pi}\int_0^{2\pi} \sin\theta \; d\theta = 0 \tag{59}$$

(c) The usual procedure of separating variables and imposing the homogeneous boundary conditions yields the general solution
$$u(x,y) = \frac{A_0}{2}y + \sum_{n\geqslant 1} A_n \cos(\pi n x)\sinh(\pi n y). \tag{60}$$
Imposing the inhomogeneous b.c. gives
$$A_0 = 1, \quad A_2\sinh(2\pi) = 2 \tag{61}$$
and the remaining A_n vanish. Hence
$$u(x,y) = \frac{y}{2} + \frac{2\cos(2\pi x)\sinh(2\pi y)}{\sinh(2\pi)}. \tag{62}$$

C Tests 2014

C.1 Midterm test

Partial Differential Equations - AUC
Midterm exam - 24/10/2014

This test contains 4 questions. Each sub-question contributes 10%.

1. Consider the linear first order PDE

$$u_x - xyu_y = 0$$

for the unknown function $u(x, y)$.

(a) Find the characteristic curves and the general solution.

(b) Find the solution corresponding to the initial data

$$u(0, y) = y^2.$$

2. Consider the wave equation

$$u_{tt} = 4u_{xx}$$

for the unknown function $u(x, t)$.

(a) Find the solution to the initial value problem on the real line $-\infty < x < \infty$ for the initial data

$$u(x, 0) = e^{-\frac{x^2}{2}}, \qquad u_t(x, 0) = -2xe^{-\frac{x^2}{2}}.$$

(b) Find the solution to the initial value problem on the half line $0 < x < \infty$ for the initial data

$$u(x, 0) = e^{-\frac{x^2}{2}}, \qquad u_t(x, 0) = 0$$

with Neumann boundary conditions at $x = 0$, i.e.

$$u_x(0, t) = 0.$$

3. Consider the function

$$\phi(x) = \begin{cases} 0 & 0 < x \leqslant l/2, \\ 1 & l/2 < x < l \end{cases}$$

on the interval $(0, l)$.

(a) Find the Fourier sine series of $\phi(x)$.

(b) Does the Fourier sine series converge pointwise on the real line ? To which function ?

4. Consider the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x), & 0 < x < \pi, \\ X(0) = X'(0), \\ X(\pi) = X'(\pi). \end{cases}$$

(a) Show that the boundary conditions above are symmetric. Write down the inner product and state the orthogonality property of the eigenfunctions X_n corresponding to the eigenvalues λ_n.

(b) Find the positive (or zero) eigenvalues and the associated eigenfunctions.

(c) Find the unique negative eigenvalue and the associated eigenfunction.

(d) Let

$$f(x) = x^3 + (3 - \pi)(x^2 + 2x + 2).$$

Does the associated general Fourier series $\sum_n A_n X_n(x)$ converge to $f(x)$ in uniform, pointwise, mean-square sense on $[0, \pi]$ and why ?

C.2 Final test

Partial Differential Equations - AUC

Final exam - 16/12/2014

This test contains 8 (sub-)questions. Each sub-question contributes 12.5%.

1. (a) Using the solution formula for the heat equation on the line, write the solution $u(x,t)$ of the initial value problem

$$\begin{cases} u_t = k u_{xx}, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = \begin{cases} a, & |x| < 1, \\ 0, & |x| > 1 \end{cases} \end{cases} \tag{1}$$

in terms of Erf function.

(b) Consider the initial value problem for the heat equation on the interval $0 < x < a$ with Dirichlet boundary conditions. Prove that the energy of the solution $u(x,t)$ is decreasing in time.

2. (a) Compute the Fourier transform of

$$f(x) = \mathrm{sign}(x) e^{-|x|}. \tag{2}$$

(b) Compute the Fourier transform of $f(cx)$ for a constant $c > 0$, in terms of the Fourier transform $\hat{f}(k)$ of the function $f(x)$.

3. (a) Let $u(x,y)$ be a harmonic function on a connected bounded open subset $D \subset \mathbb{R}^2$ that is continuous on the closure $\bar{D} = D \cup \partial D$. State the weak and strong maximum principles.

(b) Prove the weak maximum principle, by considering the function

$$v(x, y) = u(x, y) + \epsilon(x^2 + y^2) \tag{3}$$

for $\epsilon > 0$.

(c) Find the harmonic functions $u(r, \theta)$ on the circle of radius $a > 0$ that are of the form

$$u(r, \theta) = R(r)\Theta(\theta) \tag{4}$$

where (r, θ) are polar coordinates.

4. Let $y(t)$ a function satisfying the ODE

$$y''(t) + \omega^2 y(t) = 0 \tag{5}$$

and let $Y(s)$ be the Laplace transform of $y(t)$. By taking the Laplace transform of (5), obtain an equation for $Y(s)$ in terms of $y(0)$, $y'(0)$. By considering the solution $y(t) = \cos \omega t$ of (5), find the Laplace transform of $\cos \omega t$.

D Solutions Tests 2014

D.1 Solutions midterm test

1. (a) Let $y(x)$ be a characteristic curve. Then it satisfies

$$y_x + xy = 0. \tag{1}$$

This is solved by separation of variables, obtaining

$$y(x) = ce^{-x^2/2}. \tag{2}$$

The general solution is then

$$u(x, y) = f(ye^{x^2/2}). \tag{3}$$

(b) Imposing the initial condition we get

$$f(y) = y^2 \tag{4}$$

hence the solution is

$$u(x, y) = y^2 e^{x^2}. \tag{5}$$

2. (a) A simple application of d'Alembert formula gives

$$u(x, t) = e^{-(x+2t)^2/2}. \tag{6}$$

(b) The initial value functions $\phi = e^{-x^2/2}$ and $\psi = 0$ are even functions. This implies that the d'Alembert formula gives a solution which is even for all t, hence solves the i.v.p. with Neumann b.c. at $x = 0$. Substituting in d'Alembert formula we get

$$u(x, t) = \frac{1}{2}\left(e^{-(x+2t)^2/2} + e^{-(x-2t)^2/2}\right). \tag{7}$$

3. (a) The Fourier sine series coefficients are given by usual formula which in this case reads

$$A_n = \frac{2}{l}\int_{l/2}^{l} \sin\left(\frac{n\pi x}{l}\right)\, dx = -\frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right). \tag{8}$$

(b) We can apply the theorem for pointwise convergence of the full Fourier series of a periodic function $f(x)$ on the real line which is piecewise continuous and with piecewise continuous first derivative. It states that the associated Fourier series converges pointwise to

$$\frac{f(x^+) + f(x^-)}{2}. \tag{9}$$

The Fourier sine series of $f(x)$ coincides with the full Fourier series of the periodic odd extension of $f(x)$.

Hence in the present case the Fourier series converges pointwise to the function on the real line

$$\begin{cases} -1 & -l < x < -l/2, \\ -1/2 & x = -l/2, \\ 0 & -l/2 < x < l/2, \\ 1/2 & x = l/2, \\ 1 & l/2 < x < l, \\ 0 & x = l \end{cases} \tag{10}$$

extended by $2l$ periodicity outside $-l < x \leqslant l$.

4. (a) For any pair of functions f, g that satisfy the b.c. in the exercise the quantity

$$(f'g - g'f)_0^\pi \tag{11}$$

is zero, hence the b.c. are symmetric. This implies that all the eigenfunctions corresponding to different eigenvalues are orthogonal w.r.t. the inner product

$$(f, g) = \int_0^\pi f(x)g(x)\, dx, \tag{12}$$

i.e.

$$(X_n, X_m) = 0 \text{ for } \lambda_n \neq \lambda_m. \tag{13}$$

(b) For $\lambda > 0$ the general solution of the second order linear ODE is

$$X = A\cos\beta x + B\sin\beta x \tag{14}$$

for $\beta^2 = \lambda$. Imposing the b.c. we get

$$A = B \tag{15}$$

and

$$B(1+\beta^2)\sin\beta\pi = 0. \tag{16}$$

Hence the eigenvalues are

$$\lambda_n = n^2 \tag{17}$$

and the eigenfunctions

$$X_n = n\cos nx + \sin nx \tag{18}$$

for $n \geqslant 1$.
For $\lambda = 0$ we get that the general solution is

$$X = Ax + B \tag{19}$$

and the b.c. imply that $A = B = 0$ hence $\lambda = 0$ is not an e.v.

(c) For $\lambda < 0$ the general solution of the ODE is

$$X = Ae^{\beta x} + Be^{-\beta x} \tag{20}$$

for $\beta^2 = -\lambda$. Imposing the b.c. we get two equations

$$A(1-\beta) + B(1+\beta) = 0 \tag{21}$$

and

$$A(1-\beta)e^{\beta\pi} + B(1+\beta)e^{-\beta\pi} = 0. \tag{22}$$

For $\beta \neq 1$ we can substitute the first in the second and we get

$$e^{\beta\pi} - e^{-\beta\pi} = 0 \tag{23}$$

which is impossible. However for $\beta = 1$ the first equation implies $B = 0$ and so there is a nontrivial eigenfunction

$$X = e^x \tag{24}$$

corresponding to the eigenvalue $\lambda = -1$.

(d) The function $f(x)$ is a polynomial, so it is continuous and $f'(x)$, $f''(x)$ are continuous too.
Moreover we can check that it satisfies the b.c. above.
By the theorem on convergence of the general Fourier series, we know that the associated Fourier series converges uniformly to $f(x)$ on the interval $[0, \pi]$ and consequently also pointwise and in mean-square sense.
Mean-square convergence also follows from the fact that a continuous function on the interval $[0, \pi]$ has finite L^2 norm.

D.2 Solutions final test

1. (a) The solution formula gives

$$u(x,t) = \frac{a}{\sqrt{4\pi kt}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4kt}}\, dy \tag{25}$$

changing the variable of integration to p by

$$p\sqrt{4kt} = x - y \tag{26}$$

and splitting the integral in two we get easily

$$u(x,t) = \frac{a}{2}\,\mathrm{Erf}\left(\frac{x+1}{\sqrt{4kt}}\right) - \frac{a}{2}\,\mathrm{Erf}\left(\frac{x-1}{\sqrt{4kt}}\right). \tag{27}$$

(b) The energy is

$$E = \frac{1}{2}\int_0^a u^2\, dx. \tag{28}$$

Derivating w.r.t. t and using the heat equation $u_t = ku_{xx}$ we get

$$E_t = k\int_0^a uu_{xx}\, dx. \tag{29}$$

Integrating by parts, since the boundary terms vanish because of the Dirichlet b.c., this equals

$$-k\int_0^a (u_x)^2\ dx \leqslant 0. \tag{30}$$

2. (a) Splitting the integral in the definition of Fourier transform in two we get

$$\hat{f}(k) = \int_0^\infty e^{-x-ikx}\ dx - \int_{-\infty}^0 e^{x-ikx}\ dx. \tag{31}$$

   Integrating the exponentials we get

$$\hat{f}(k) = \frac{-2ik}{1+k^2}. \tag{32}$$

   (b) Just a change of variable in the definition of Fourier transform.

3. (a) Weak maximum principle: the maximum of u is achieved on ∂D. Strong maximum principle: for u non-constant, the maximum of u is achieved on ∂D only.

   (b) See book.

   (c) See book.

4. Apply the Laplace transform and get

$$Y(s) = \frac{sy(0) + y'(0)}{s^2+\omega^2}. \tag{33}$$

   If $y(t) = \cos\omega t$ then $y(0) = 1$ and $y'(0) = 0$ so

$$Y(s) = \frac{s}{s^2+\omega^2}. \tag{34}$$

E Tests 2015

E.1 First test

Partial Differential Equations - AUC
First exam - 25/09/2015

1. Find the general solution of
$$\frac{dy}{dx} - \frac{2}{x^3}y = \frac{3}{x^3}$$
by finding the integrating factor $I(x, y)$ and solving the resulting exact equation.

2. Let $\mathcal{L}$ be a linear operator. Let u_0 be a solution of the linear homogeneous equation $\mathcal{L}(u) = 0$ and let u_i $(i = 1, 2)$ be a solution of the linear inhomogeneous equation
$$\mathcal{L}(u) = g_i.$$
Show that for any constants c_1, c_2 the function $u_0 + c_1u_1 + c_2u_2$ is a solution of the inhomogeneous equation
$$\mathcal{L}(u) = c_1g_1 + c_2g_2.$$

3. Consider the PDE
$$u_x + (\cos x)u_y = 0 \tag{1}$$
for the unknown function $u(x, y)$.
   (a) Find the characteristic curves in the xy-plane.
   (b) Write down the general solution in terms of an arbitrary function of one variable f.
   (c) Check directly that such solution satisfies the equation (1).
   (d) Find the solution $u(x, y)$ that satisfies the auxiliary condition
   $$u(\frac{\pi}{2}, y) = y^3.$$
   (e) Verify that the function $u(x, y) = x \sin y$ is a particular solution of the inhomogeneous equation
   $$u_x + (\cos x)u_y = \sin y + x \cos x \cos y. \tag{2}$$
   Then write down the general solution of the inhomogeneous equation (5).

4. Write down the simple transport equation describing the concentration $w(x, t)$ of a pollutant in a pipe containing a liquid moving to the right with constant speed v. If at time $t = 0$ the concentration is given by $w(x, 0) = \frac{1}{1+x^8}$, what is the concentration at time $t = T$?

5. Find the solution of the wave equation $u_{tt} = 9u_{xx}$ with initial conditions
   (a) $u(x, 0) = \frac{x}{1+x^4}, \quad u_t(x, 0) = 0$;
   (b) $u(x, 0) = 0, \quad u_t(x, 0) = x^2e^{-x^3}$.

E.2 First test (modified)

Partial Differential Equations - AUC
First exam (modified) - 06/10/2015

1. Find the general solution of
$$\frac{dy}{dx} - \frac{4}{x^5}y = \frac{12}{x^5}$$
by finding the integrating factor $I(x,y)$ and solving the resulting exact equation.

2. Let $\mathcal{L}$ be a linear operator. Let u_i $(i = 1, 2)$ be a solution of the linear inhomogeneous equation
$$\mathcal{L}(u) = c_i,$$
where c_1 and c_2 are *constants*. Show that the function $u = c_2u_1 - c_1u_2$ is a solution of the *homogeneous* equation
$$\mathcal{L}(u) = 0.$$

3. Consider the PDE
$$u_x + \frac{u_y}{x} = 0 \tag{1}$$
for the unknown function $u(x,y)$ defined in the region $U = \{(x,y)|x > 0\}$ in the xy-plane.
   (a) Find the characteristic curves in the region U.
   (b) Write down the general solution in terms of an arbitrary function of one variable f.
   (c) Check directly that such solution satisfies the equation (1).
   (d) Find the solution $u(x,y)$ that satisfies the auxiliary condition
   $$u(1,y) = e^{-y}.$$
   (e) Verify that the function $u(x,y) = xe^y$ is a particular solution of the inhomogeneous equation
   $$u_x + \frac{u_y}{x} = 2e^y. \tag{2}$$
   Then write down the general solution of the inhomogeneous equation (5).

4. Write down the simple transport equation describing the concentration $w(x,t)$ of a pollutant in a pipe containing a liquid moving to the left with constant speed v. If at time $t = 0$ the concentration is given by $w(x,0) = \frac{e^{-x^4}}{1+2x^2}$, what is the concentration at time $t = T$?

5. Find the solution of the wave equation $u_{tt} = \frac{1}{4}u_{xx}$ with initial conditions
   (a) $u(x,0) = e^{-x^2}\cos x, \quad u_t(x,0) = 0$;
   (b) $u(x,0) = 0, \quad u_t(x,0) = \frac{1}{1+x^2}$.

E.3 Second test

Partial Differential Equations - AUC

Second exam - 23/10/2015

1. Consider the initial value problem for the wave equation on the *negative* half-line $-\infty < x < 0$ with Dirichlet boundary conditions at $x = 0$:

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0 \end{cases} \tag{1}$$

   (a) Define odd functions ϕ_o, ψ_o that extend ϕ, ψ to the whole real line and show that the solution $u(x,t)$, given by the d'Alembert formula with initial data ϕ_o, ψ_o, is also odd as a function of x.

   (b) Show that such a solution, when restricted to the half line $x < 0$, solves the i.v.p. (1).

   (c) Write the formula for $u(x,t)$ in terms of ϕ and ψ in the case $-x > ct > 0$, and sketch the dependence domain for a point (x_0, t_0) with $-x_0 > ct_0 > 0$.

   (d) Write the formula for $u(x,t)$ in terms of ϕ and ψ in the case $0 < -x < ct$, and sketch the dependence domain for a point (x_0, t_0) with $0 < -x_0 < ct_0$.

   (e) Sketch the influence domain of an interval $[a,b]$ for $a < b < 0$.

2. For a solution $u(x,t)$ of the wave equation on the interval $[0,l]$ (describing a string of linear mass density ρ and tension T) satisfying the mixed boundary condition $u(0,t) = 0 = u_x(l,t)$ (Dirichlet at one end, Neumann at the other), prove that the total energy

$$E = \frac{1}{2}\int_0^l (\rho u_t^2 + T u_x^2)dx$$

   is conserved.

3. Compute:

   (a) The Fourier sine series of $\phi(x) = x - \frac{\pi}{2}$ on the interval $(0,\pi)$.

   (b) The Fourier cosine series of $\phi(x) = x - \frac{\pi}{2}$ on the interval $(0,\pi)$.

   (c) The full Fourier series of the periodic function ϕ of period 2π defined by

$$\phi(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \leqslant x < \pi \end{cases} \tag{2}$$

   and extended periodically.

4. Recall that a sequence of functions $\{f_n\}$, $n = 1, 2, \ldots$, defined on the interval $[0, 1]$, converges to 0 *in the mean* if the sequence of L^1-norms

$$\|f_n\|_1 = \int_0^1 |f_n(x)|dx$$

tends to 0 as $n \to \infty$; it converges to 0 *in the mean-squared* if the sequence of L^2-norms

$$\|f_n\|_2 = \left(\int_0^1 |f_n(x)|^2 dx\right)^{\frac{1}{2}}$$

tends to 0 as $n \to \infty$.

Consider the sequence defined as follows:

$$f_n(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{4n^2} \\ n & \frac{1}{4n^2} \leqslant x \leqslant \frac{1}{2n^2} \\ 0 & \frac{1}{2n^2} < x \leqslant 1. \end{cases}$$

   (a) Show that the sequence $\{f_n\}$ converges to the zero function pointwise on $[0, 1]$;

   (b) Show that $\{f_n\}$ converges to 0 in the mean;

   (c) Show that $\{f_n\}$ does *not* converge to 0 in the mean-squared;

   (d) Conclude that $\{f_n\}$ does not converge uniformly.

E.4 Second test (modified)

Partial Differential Equations - AUC

Second exam (modified) - 02/11/2015

1. Consider the initial boundary value problem for the wave equation on the *negative* half-line $-\infty < x < 0$ with Neumann boundary conditions at $x = 0$:

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \\ u_x(0, t) = 0 \end{cases} \tag{1}$$

   (a) Define even functions ϕ_e, ψ_e that extend ϕ, ψ to the whole real line and show that the solution $u(x, t)$, given by the d'Alembert formula with initial data ϕ_e, ψ_e, is also even as a function of x.

   (b) Show that such a solution, when restricted to the half line $x < 0$, solves the i.b.v.p. (1).

   (c) Write the formula for $u(x, t)$ in terms of ϕ and ψ in the case $-x > ct > 0$, and sketch the dependence domain for a point (x_0, t_0) with $-x_0 > ct_0 > 0$.

(d) Write the formula for $u(x,t)$ in terms of ϕ and ψ in the case $0 < -x < ct$, and sketch the dependence domain for a point (x_0, t_0) with $0 < -x_0 < ct_0$.

(e) Sketch the influence domain of an interval $[a,b]$ for $a < b < 0$.

2. For a solution $u(x,t)$ of the wave equation on the interval $[0,l]$ (describing a string of linear mass density ρ and tension T) satisfying the mixed boundary condition $u_x(0,t) = 0 = u(l,t)$ (Neumann at one end, Dirichlet at the other), prove that the total energy

$$E = \frac{1}{2}\int_0^l (\rho u_t^2 + T u_x^2)dx$$

is conserved.

3. Compute:

(a) The Fourier sine series of $\phi(x) = \pi - x$ on the interval $(0,\pi)$.

(b) The Fourier cosine series of $\phi(x) = \pi - x$ on the interval $(0,\pi)$.

(c) The full Fourier series of the periodic function ϕ of period 2π defined by

$$\phi(x) = \begin{cases} -x & -\pi < x < 0 \\ 0 & 0 \leqslant x < \pi \end{cases}$$

and extended periodically.

4. Recall that a sequence of functions $\{f_n\}$, $n = 1, 2, \ldots$, defined on the interval $[0,1]$, converges to 0 *in the mean* if the sequence of L^1-norms

$$\|f_n\|_1 = \int_0^1 |f_n(x)|dx$$

tends to 0 as $n \to \infty$; it converges to 0 *in the mean-square* if the sequence of L^2-norms

$$\|f_n\|_2 = \left(\int_0^1 |f_n(x)|^2 dx\right)^{\frac{1}{2}}$$

tends to 0 as $n \to \infty$.

Consider the sequence defined as follows:

$$f_n(x) = \begin{cases} 0 & 0 \leqslant x < 1 - \frac{1}{2n^2} \\ n & 1 - \frac{1}{2n^2} \leqslant x \leqslant 1 - \frac{1}{4n^2} \\ 0 & 1 - \frac{1}{4n^2} < x \leqslant 1. \end{cases}$$

(a) Show that the sequence $\{f_n\}$ converges to the zero function pointwise on $[0,1]$;

(b) Show that $\{f_n\}$ converges to 0 in the mean;

(c) Show that $\{f_n\}$ does *not* converge to 0 in the mean-square;

(d) Conclude that $\{f_n\}$ does not converge uniformly.

E.5 Third test

Partial Differential Equations - AUC

Third exam - 20/11/2015

1. Let $u(x,t)$ be a solution to the heat equation on the rectangle R given by

$$0 \leqslant x \leqslant 3, \quad 0 \leqslant t \leqslant 2.$$

Let U be the part of the boundary of R consisting of the bottom ($t = 0$) and the two lateral sides ($x = 0$ and $x = 3$).

   (a) Suppose it is known that $u(x,t) \leqslant 27$ for all $(x,t) \in U$. What can you say about $u(1,1)$ according to the *weak* maximum principle?

   (b) Suppose it is further known that $u(2,1) = 27$. What can you say about $u(1,1)$ according to the *strong* maximum principle?

2. Consider the heat equation $u_t = ku_{xx}$ on the interval $(0,\pi)$ with mixed boundary conditions $u_x(0,t) = u(\pi,t) = 0$. Find the separated solutions (you may assume the positivity of the eigenvalues of $-\frac{d^2}{dx^2}$ for these b.c.).

3. Let $u(x,t)$ be the solution of the initial value problem for the heat equation on the real line

$$\begin{cases} u_t = 4u_{xx}, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

   (a) Write down the solution formula (Poisson integral) for $u(x,t)$.

   (b) Suppose $\phi(x)$ has a jump discontinuity at $x = 0$. Does it follow that $u(x,t)$ has a discontinuity at $(x,t) = (2,1)$? What about $\frac{\partial^3 u}{\partial t^3}(x,t)$?

   (c) Compute explicitly (in terms of the error function) the solution for the initial data

$$\phi(x) = \begin{cases} -2 & x < 0 \\ 3 & x > 0. \end{cases}$$

4. (a) Compute the Fourier transform $\hat{f}(p)$ of

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) If $\hat{f}(p)$ is the Fourier transform of $f(x)$, show that the Fourier transform of $f(cx)$, $c > 0$, is $\frac{1}{c}\hat{f}(\frac{p}{c})$.

   (c) Apply the Fourier transform (in the x variable) to the following initial value problem:

$$\begin{cases} u_t = \frac{1}{2}u_{xx} - u, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

and solve the initial value problem for the resulting ODE. Conclude that the solution $u(x,t)$ of the original IVP is given by the convolution of $\phi(x)$ with the inverse Fourier transform $\psi(x,t)$ of the function $\hat{\psi}(p,t) = e^{-(\frac{p^2}{2}+1)t} = e^{-t}e^{-\frac{p^2}{2}t}$.

(d) Write down the formula for the Fourier transform of the Gaussian function $e^{-\frac{x^2}{2}}$. Use this formula, together with part (b) and the linearity of the Fourier transform, to find $\psi(x,t)$. Then write down the solution formula for the original IVP (1).

E.6 Third test (modified)

Partial Differential Equations - AUC

Third exam (modified) - 26/11/2015

1. Let $u(x,t)$ be a solution to the heat equation on the rectangle R given by

$$0 \leqslant x \leqslant 3, \quad 0 \leqslant t \leqslant 2.$$

Let U be the part of the boundary of R consisting of the bottom ($t=0$) and the two lateral sides ($x=0$ and $x=3$).

(a) Suppose it is known that $u(x,t) \geqslant 11$ for all $(x,t) \in U$. What can you say about $u(1,1)$ according to the *weak* maximum principle?

(b) Suppose it is further known that $u(2,1) = 11$. What can you say about $u(1,1)$ according to the *strong* maximum principle?

2. Consider the heat equation $u_t = ku_{xx}$ on the interval $(0,\pi)$ with mixed boundary conditions $u(0,t) = u_x(\pi,t) = 0$. Find the separated solutions (you may assume the positivity of the eigenvalues of $-\frac{d^2}{dx^2}$ for these b.c.).

3. Let $u(x,t)$ be the solution of the initial value problem for the heat equation on the real line

$$\begin{cases} u_t = 9u_{xx}, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

(a) Write down the solution formula (Poisson integral) for $u(x,t)$.

(b) Suppose $\phi(x)$ has a jump discontinuity at $x=0$. Does it follow that $u(x,t)$ has a discontinuity at $(x,t) = (3,1)$? What about $\frac{\partial^3 u}{\partial x \partial t^2}(x,t)$?

(c) Compute explicitly (in terms of the error function) the solution for the initial data

$$\phi(x) = \begin{cases} 3 & x < 0 \\ 2 & x > 0. \end{cases}$$

4. (a) Compute the Fourier transform $\hat{f}(p)$ of

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $\hat{f}(p)$ is the Fourier transform of $f(x)$, show that the Fourier transform of $f(\frac{x}{c})$, $c > 0$, is $c\hat{f}(cp)$.

(c) Apply the Fourier transform (in the x variable) to the following initial value problem:

$$\begin{cases} u_t = \frac{1}{2}u_{xx} + u, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

and solve the initial value problem for the resulting ODE. Conclude that the solution $u(x,t)$ of the original IVP is given by the convolution of $\phi(x)$ with the inverse Fourier transform $\psi(x,t)$ of the function $\hat{\psi}(p,t) = e^{-(\frac{p^2}{2}-1)t} = e^t e^{-\frac{p^2}{2}t}$.

(d) Write down the formula for the Fourier transform of the Gaussian function $e^{-\frac{x^2}{2}}$. Use this formula, together with part (b) and the linearity of the Fourier transform, to find $\psi(x,t)$. Then write down the solution formula for the original IVP (1).

E.7 Fourth test

Partial Differential Equations - AUC

Fourth exam - 18/12/2015

In this test there are 3 questions. Each sub-question contributes 10%.

1. (a) If the Laplace transform of $f(t)$ is given by $F(s)$, derive the formula for the Laplace transform of $f'(t)$.

   (b) The Laplace transform of $\cosh t$ is $\frac{s}{s^2-1}$. Use the formula derived in part (a) to obtain the Laplace transform of $\sinh t$.

   (c) Using the Laplace transform method, solve the initial value problem

$$\frac{dy}{dt} + 2y = \sinh t, \quad y(0) = 0.$$

2. Find the regions in the xy plane where the equation

$$yu_{xx} + 2xu_{xy} + u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

3. (a) Let u be a harmonic function on the unit disk

$$D = \{\mathbf{x} = (x,y) \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}$$

whose boundary value is

$$u_{|r=1} = 1 + \sin\theta - \cos 2\theta,$$

where (r,θ) are the polar coordinates. What is $u(\mathbf{0})$?

(b) Find the harmonic function $u(x,y)$ on the square

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1, 0 < y < 1\}$$

such that

$$u_x(0,y) = u_x(1,y) = u(x,1) = 0, \quad u(x,0) = 1.$$

(c) Consider the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{H}, \\ u = h & \text{on } \partial\mathbb{H}, \end{cases} \tag{1}$$

where $\mathbb{H} = \{\mathbf{x} = (x,y) | y > 0\}$ is the upper half-plane, so $\partial\mathbb{H}$ is the x-axis; notice that $\mathbb{H}$ is *unbounded*. Show that the function $u(x,y) = xy$ is harmonic on $\mathbb{H}$. What is its boundary value h? Is u a unique solution of the BVP (1) with this h?

(d) Show that the solutions of the BVP (1) satisfying the additional condition

$$\lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0$$

are unique (*Hint:* apply the maximum principle to the half-disk $H_R = \{\mathbf{x} = (x,y) | y > 0, |\mathbf{x}| < R\}$ and then let $R \to \infty$).

(e) Recall that the *fundamental solution* $F(\mathbf{x}) = \frac{1}{2\pi} \log |\mathbf{x}|$ of the Laplace equation in two dimensions satisfies

$$\Delta F(\mathbf{x}) = \delta(\mathbf{x}).$$

Given a point $\mathbf{x} = (x,y) \in \mathbb{H}$, let $\mathbf{x}^* = (x,-y)$ be its reflection in the x-axis. Show that for all $\mathbf{x} = (x,0) \in \partial\mathbb{H}$ and all $\mathbf{x_0} = (x_0,y_0) \in \mathbb{H}$ we have $|\mathbf{x} - \mathbf{x_0}| = |\mathbf{x} - \mathbf{x_0^*}|$. Conclude that

$$G(\mathbf{x}, \mathbf{x_0}) = F(\mathbf{x} - \mathbf{x_0}) - F(\mathbf{x} - \mathbf{x_0^*})$$

is the Green's function for $\mathbb{H}$.

(f) Use the Green's function obtained in the previous part to show that the general solution formula

$$u(\mathbf{x_0}) = \int_{\partial\mathbb{H}} h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x_0})}{\partial \mathbf{n}} ds$$

for the BVP (1) reduces to

$$u(x_0,y_0) = \frac{y_0}{\pi} \int_{-\infty}^{+\infty} \frac{h(x)dx}{(x-x_0)^2 + y_0^2}$$

(*Hint:* observe that $\frac{\partial}{\partial \mathbf{n}} = -\left.\frac{\partial}{\partial y}\right|_{y=0}$ in this case).

F Solutions Tests 2015

F.1 First test

1. The integrating factor is
$$I = e^{x^{-2}}. \tag{1}$$
Multiplication by I gives an exact equation that we can integrate in the usual way obtaining
$$e^{x^{-2}}(2y+3) = c \tag{2}$$
or explicitly
$$y = -\frac{3}{2} + ce^{-x^{-2}}. \tag{3}$$
Alternatively, one can notice that the original equation can be re-written in a separated form as
$$\frac{dx}{x^3} = \frac{dy}{2y+3} \tag{4}$$
and arrive at the same solution.

2. Just calculate $\mathcal{L}(u_0 + c_1 u_1 + c_2 u_2)$ using linearity.

3. (a) Suppose $y(x)$ is the equation for a characteristic curve, then
$$0 = \frac{du(x, y(x))}{dx} = u_x + u_y y_x = u_y(y_x - \cos x), \tag{5}$$
hence
$$y_x = \cos x \tag{6}$$
so that the characteristic curves have equations
$$y = \sin x + c. \tag{7}$$
(b)
$$u(x, y) = f(y - \sin x) \tag{8}$$
(c) Trivial.
(d) Imposing the initial condition in the general solution above
$$u(\frac{\pi}{2}, y) = y^3 = f(y - \sin\frac{\pi}{2}) = f(y-1). \tag{9}$$
By a change of variable $z = y - 1$ we get
$$f(z) = (z+1)^3 \tag{10}$$
and finally the solution
$$u(x, t) = (y - \sin x + 1)^3. \tag{11}$$
(e) The verification is trivial. Then by the principle of superposition the general solution is
$$u(x, y) = f(y - \sin x) + x \sin y. \tag{12}$$

4. The transport equation for motion to the right is

$$w_t + v w_x = 0. \tag{13}$$

The general solution is

$$w(x,t) = f(x - vt). \tag{14}$$

Imposing the initial condition we get

$$w(x,0) = \frac{1}{1+x^8} = f(x), \tag{15}$$

and finally

$$w(x,t) = \frac{1}{1+(x-vt)^8}. \tag{16}$$

5. (a) Using d'Alembert formula with $c^2 = 9$, $\psi = 0$, $\phi(x) = \frac{x}{1+x^4}$ we have

$$u(x,t) = \frac{1}{2}\left(\frac{x-3t}{1+(x-3t)^4} + \frac{x+3t}{1+(x+3t)^4}\right). \tag{17}$$

   (b) Again by d'Alembert formula

$$u(x,t) = \frac{1}{18}\left(e^{-(x-3t)^3} - e^{-(x+3t)^3}\right). \tag{18}$$

F.2 Second test

1. (a) The odd extension is given by

$$\phi_o(x) = \begin{cases} \phi(x) & x < 0 \\ -\phi(-x) & x > 0, \end{cases}$$

   and similarly for ψ. The solution is then given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}(\phi_o(x+ct) + \phi_o(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_o(s)\ ds.$$

   One can either see that u is odd directly from this formula, or by observing that $-u(-x,t)$ solves the same i.b.v.p as $u(x,t)$, and invoking uniqueness.

   (b) The restriction of $u(x,t)$ to $x < 0$ still solves the wave equation with the correct initial data. Moreover by the previous part u is odd in x at all t, hence the b.c. is satisfied.

   (c) For $-x > ct > 0$, we have both $x + ct < 0$ and $x - ct < 0$, hence the d'Alembert formula is simply

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds.$$

(d) For $0 < -x < ct$, we now have $x + ct > 0$, while $x - ct < 0$ so the formula becomes

$$u(x,t) = \frac{1}{2}(\phi(x-ct) - \phi(-x-ct)) + \frac{1}{2c}\int_{x-ct}^{-x-ct} \psi(s)\ ds.$$

(e) The sketches in all cases are determined by the intervals of integration in the above formulas.

2. The same calculation as the one in the textbook for the unbounded case, except it now involves the boundary term

$$Tu_t u_x|_0^l = T(u_t(l,t)u_x(l,t)) - u_t(0,t)u_x(0,t))$$

which vanishes because the boundary conditions $u(0,t) = 0 = u_x(l,t)$ imply also that $u_t(0,t) = 0$.

3. (a) The Fourier sine series coefficients B_n, $n > 0$, are given by

$$B_n = \frac{2}{\pi}\int_0^\pi \left(x - \frac{\pi}{2}\right)\sin nx\ dx = \begin{cases} -\frac{2}{n} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Hence the series is

$$\phi(x) = -\sum_{m=1}^\infty \frac{\sin 2mx}{m}.$$

(b) The Fourier cosine series coefficients A_n, $n \geqslant 0$, are given by

$$A_n = \frac{2}{\pi}\int_0^\pi \left(x - \frac{\pi}{2}\right)\cos nx\ dx = \begin{cases} -\frac{4}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(in particular, $A_0 = 0$). Hence the series is

$$\phi(x) = -\frac{4}{\pi}\sum_{m=1}^\infty \frac{\cos(2m-1)x}{(2m-1)^2}.$$

(c) The result is

$$\phi(x) = \frac{\pi}{4} - \sum_{n=1}^\infty \left(\frac{2}{\pi(2n-1)^2}\cos(2n-1)x + \frac{(-1)^n}{n}\sin nx\right).$$

4. For this question it helps to draw the graph of f_n for the first few values of n in order to see the trend.

(a) We have $f_n(0) = 0$ for all n, while for each fixed $x \in (0,1]$ there exists an N such that $0 < \frac{1}{2n^2} < x$ for all $n > N$, hence $f_n(x) = 0$ for all sufficiently large n. It follows that f_n converges to the zero function on $[0,1]$.

(b) We have

$$\|f_n\|_1 = \int_0^1 |f_n(x)|dx = \int_{\frac{1}{4n^2}}^{\frac{1}{2n^2}} n\ dx = \frac{n}{4n^2} = \frac{1}{4n},$$

which clearly tends to 0, implying that the sequence converges to 0 in the mean.

(c) We have

$$\|f_n\|_2 = \left(\int_0^1 |f_n(x)|^2 dx\right)^{\frac{1}{2}} = \left(\int_{\frac{1}{4n^2}}^{\frac{1}{2n^2}} n^2\ dx\right)^{\frac{1}{2}} = \sqrt{\frac{n^2}{4n^2}} = \frac{1}{2},$$

which clearly does *not* tend to 0, hence the sequence does not converge in the mean-square.

(d) We showed in class that uniform convergence implies mean-square convergence; since the sequence does not converge in the mean-square by the previous part, it also does not converge uniformly. Alternatively, one can observe that the max-norm of f_n is n, forming a *divergent* sequence.

F.3 Third test

1. (a) The weak maximum principle implies that $u(1,1) \leqslant 27$;

   (b) The additional piece of information – that the maximum value is achieved at an interior point – implies that u is, in fact, constant; hence, $u(1,1) = 27$.

2. Studying the eigenvalue problem $-X'' = \lambda X$ with specified b.c. ($X'(0) = X(\pi) = 0$) we get the eigenvalues

$$\lambda_n = \left(n + \frac{1}{2}\right)^2, \quad n \geqslant 0$$

with eigenfunctions

$$X_n = \cos\left(n + \frac{1}{2}\right)x.$$

Solving the equation for T we get the separated solutions

$$u_n = e^{-(n+\frac{1}{2})^2 kt} \cos(n + \frac{1}{2})x$$

for $n \geqslant 0$.

3. (a) See book (just set $k = 4$).

   (b) By the smoothing property of the heat equation all the derivatives exist and are continuous for $t > 0$. Hence, the answer is "no" in both cases.

   (c) The formula for the solution of the heat equation on the line yields, after some manipulations of the integral,

$$u(x,t) = \frac{1}{2} + \frac{5}{2}\,\mathrm{Erf}\,\frac{x}{4\sqrt{t}}.$$

4. (a) Integration gives
$$\hat{f}(p) = \frac{1}{p^2}(2 - 2\cos p) = \frac{4}{p^2}\sin^2\frac{p}{2}.$$
   (b) Immediate by substitution.
   (c) By applying the Fourier transform to the equation at hand and the initial condition we get
$$\begin{cases} \hat{u}_t = -\left(\frac{1}{2}p^2 + 1\right)\hat{u}, \\ \hat{u}(p,0) = \hat{\phi}(p), \end{cases}$$
   where $\hat{u}(p,t)$ is the Fourier transform of $u(x,t)$ in x and $\hat{\phi}(p)$ is the Fourier transform of $\phi(x)$. The solution of the ODE in t is clearly
$$\hat{u}(p,t) = \hat{\phi}(p)e^{-(\frac{1}{2}p^2+1)t} = \hat{\psi}(p,t)\hat{\phi}(p) = \mathcal{F}[\psi * \phi]. \tag{19}$$
   By taking the inverse Fourier transform we conclude that $u(x,t) = \psi(x,t) * \phi(x)$.
   (d) We know that
$$\mathcal{F}[e^{-x^2/2}] = \sqrt{2\pi}e^{-p^2/2}$$
   and by the rescaling property of the Fourier transform (cf. part (b)) we get
$$\mathcal{F}[e^{-(ax)^2/2}] = \frac{\sqrt{2\pi}}{a}e^{-p^2/(2a^2)}.$$
   In this case $a = 1/\sqrt{t}$ hence, using the linearity of the (inverse) Fourier transform, we get
$$\psi(x,t) = \mathcal{F}^{-1}(e^{-(\frac{1}{2}p^2+1)t}) = e^{-t}\mathcal{F}^{-1}(e^{-\frac{1}{2}p^2t}) = \frac{e^{-t}}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}},$$
   and thus
$$u(x,t) = \psi * \phi = \frac{e^{-t}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2t}}\phi(y)\,dy.$$

F.4 Fourth test

1. (a) See book.
   (b) See book.
   (c) Taking the Laplace transform we get
$$(s+2)Y(s) = \frac{1}{s^2-1}, \tag{20}$$
   which can be rewritten as
$$Y(s) = \frac{1}{3}\frac{1}{s+2} + \frac{1}{6}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}. \tag{21}$$
   Therefore the solution is
$$y(t) = \frac{1}{3}e^{-2t} + \frac{1}{6}e^{t} - \frac{1}{2}e^{-t}. \tag{22}$$

2. We have
$$\Delta = x^2 - y, \tag{23}$$
therefore the equation is parabolic on the parabola $\Delta = 0$, elliptic above the parabola and hyperbolic otherwise.

3. (a) The value is given by the average, i.e., 1.

   (b) Separation of variables, imposing only the homogeneous boundary conditions gives
$$u = \frac{A_0}{2}(y-1) + \sum_{n \geqslant 1} \cos(n\pi x)(\sinh(\pi n)\cosh(n\pi y) - \cosh(\pi n)\sinh(n\pi y)). \tag{24}$$
   Imposing the remaining boundary condition, and recalling the cosine Fourier series of the constant function, we get that all coefficients A_n are zero unless $n = 0$. Therefore the solution is
$$u = 1 - y \tag{25}$$
   as one could have seen from the beginning!

   (c) No uniqueness, e.g.,
$$u = \alpha y + \beta xy. \tag{26}$$

   (d) Let $M(R)$ and $m(R)$ be the maximum and minimum of u in H_R. The additional condition implies that they go to zero for $R \to \infty$, therefore u has to be identically zero. Uniqueness follows.

   (e)

   (f)

G Tests 2016

G.1 First test

Partial Differential Equations - AUC
First exam - 30/09/2016

1. Let $\mathcal{L}$ be a linear operator. Let $u = c_1 u_1 + c_2 u_2 + v$ be a solution of the linear inhomogeneous equation
$$\mathcal{L}(u) = g \tag{1}$$
for each value of the constants c_1 and c_2. Show that v is a solution of the inhomogeneous equation (1), and that u_1 and u_2 are solutions of the associated homogeneous equation, i.e.,
$$\mathcal{L}(u_i) = 0, \qquad i = 1, 2.$$

2. (a) Find the general solution of
$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} - x^2 = 0, \qquad x > 0 \tag{2}$$
by reducing it to a linear first order ODE, finding the integrating factor, and solving the resulting exact equation.
   (b) From the general solution of equation (2) identify two solutions of the associated homogeneous equation and show that they are linearly independent by computing their Wronskian.

3. Consider the PDE
$$u_x + (1 + x^2)u_y = 0 \tag{3}$$
for the unknown function $u(x, y)$.
   (a) Find the characteristic curves in the xy-plane.
   (b) Write down the general solution in terms of an arbitrary function of one variable f.
   (c) Check directly that such solution satisfies the equation (3).
   (d) Find the solution $u(x, y)$ that satisfies the auxiliary condition
$$u(0, y) = e^{-\frac{1}{2}y^2}. \tag{4}$$
   (e) Find the general solution of the inhomogeneous equation
$$u_x + (1 + x^2)u_y = \cos(x). \tag{5}$$
(Hint: look for a particular solution that depends only on x.)

4. Consider the following initial value problem
$$\begin{cases} u_{tt} = 4u_{xx} \\ u(x, 0) = \sin x + e^x \\ u_t(x, 0) = -2\cos x + 2e^x. \end{cases} \tag{6}$$
   (a) Find the solution $u(x, t)$ of the initial value problem.
   (b) Identify the right-moving and left-moving parts of the solution.

G.2 Second test

Partial Differential Equations - AUC
Second exam - 28/10/2016

1. Consider the initial value problem for the wave equation on the half-line $0 < x < \infty$ with Neumann boundary conditions at $x = 0$:
$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < \infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u_x(0,t) = 0 \end{cases} \tag{1}$$
   (a) Define even functions ϕ_e, ψ_e that extend ϕ, ψ to the real line and show that the solution $u(x,t)$ given by d'Alembert formula with initial data ϕ_e, ψ_e is also even. Show that such solution, restricted on the half line $x > 0$, solves the i.v.p. (1).
   (b) Write the formula for $u(x,t)$ in terms of ϕ, ψ in the cases $x > ct$ and $x < ct$ (assume $t > 0$).
   (c) Sketch in (x,t)-plane the dependence domain for a point (x_0, t_0) with $t_0 > 0$, $0 < x_0 < ct_0$.

2. Consider a solution $u(x,t)$ to the wave equation on the interval $[0,l]$ (describing a string of length l of linear mass density ρ and tension T)
$$u_{tt} = \frac{T}{\rho} u_{xx} \tag{2}$$
   with Neumann boundary conditions, i.e.,
$$u_x(0,t) = 0, \quad u_x(l,t) = 0. \tag{3}$$
   (a) Prove energy conservation.
   (b) Using energy conservation, show that the only solution to the initial value problem with zero initial data, i.e., $u(x,0) = 0$, $u_t(x,0) = 0$ for $0 \leqslant x \leqslant l$, is the solution $u(x,t) \equiv 0$.

3. Consider the wave equation on the interval with Dirichlet boundary conditions
$$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < l, \\ u(0,t) = u(l,t) = 0. \end{cases} \tag{4}$$
   (a) Find the separated solutions $u_n = X_n(x) T_n(t)$, $n \geqslant 1$. In particular solve the associated eigenvalue problem, assuming the eigenvalue λ is positive, and solve the associated equation for T.
   (b) Write down the general solution in terms of an infinite series and find the solution of the initial value problem with initial conditions
$$\begin{cases} u(x,0) = \sin(\frac{\pi}{l} x), \\ u_t(x,0) = \frac{2\pi}{l} \sin(\frac{2\pi}{l} x). \end{cases} \tag{5}$$

4. Consider the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x), \quad 0 < x < 1, \\ X(0) = 0, \\ X'(1) = X(1). \end{cases} \tag{6}$$

   (a) Show that $\lambda = 0$ is an eigenvalue and compute the corresponding eigenfunction.

   (b) Let $\gamma > 0$ be any positive number that satisfies the equation

$$\gamma = \tan(\gamma). \tag{7}$$

   Show that $X(x) = \sin(\gamma x)$ is an eigenfunction corresponding to the eigenvalue $\lambda = \gamma^2$.

   (c) Using Green's second identity prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

G.3 Second test (modified)

Partial Differential Equations - AUC

Second exam (modified) - 31/10/2016

1. Consider the initial value problem for the wave equation on the half-line $0 < x < \infty$ with Dirichlet boundary conditions at $x = 0$:

$$\begin{cases} u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0 \end{cases} \tag{1}$$

   (a) Define odd functions ϕ_o, ψ_o that extend ϕ, ψ to the real line and show that the solution $u(x,t)$ given by d'Alembert formula with initial data ϕ_o, ψ_o is also odd. Show that such solution, restricted on the half line $x > 0$, solves the i.v.p. (1).

   (b) Write the formula for $u(x,t)$ in terms of ϕ, ψ in the cases $x > ct$ and $x < ct$ (assume $t > 0$).

   (c) Sketch in (x,t)-plane the dependence domain for a point (x_0, t_0) with $t_0 > 0$, $0 < x_0 < ct_0$.

2. Consider a solution $u(x,t)$ to the wave equation on the interval $[0,l]$ (describing a string of length l of linear mass density ρ and tension T)

$$u_{tt} = \frac{T}{\rho} u_{xx} \tag{2}$$

   with Neumann boundary conditions, i.e.,

$$u_x(0,t) = 0, \quad u_x(l,t) = 0. \tag{3}$$

(a) Prove energy conservation.

(b) Using energy conservation, show that the only solution to the initial value problem with zero initial data, i.e., $u(x,0) = 0$, $u_t(x,0) = 0$ for $0 \leqslant x \leqslant l$, is the solution $u(x,t) \equiv 0$.

3. Consider the wave equation on the interval with Dirichlet boundary conditions

$$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < l, \\ u(0,t) = u(l,t) = 0. \end{cases} \tag{4}$$

(a) Find the separated solutions $u_n = X_n(x)T_n(t)$, $n \geqslant 1$. In particular solve the associated eigenvalue problem, assuming the eigenvalue λ is positive, and solve the associated equation for T.

(b) Write down the general solution in terms of an infinite series and find the solution of the initial value problem with initial conditions

$$\begin{cases} u(x,0) = \sin(2\frac{\pi}{l}x), \\ u_t(x,0) = \frac{\pi}{l}\sin(\frac{\pi}{l}x). \end{cases} \tag{5}$$

4. Consider the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x), \quad 0 < x < 1, \\ X(0) = 0, \\ X'(1) = X(1). \end{cases} \tag{6}$$

(a) Show that $\lambda = 0$ is an eigenvalue and compute the corresponding eigenfunction.

(b) Let $\gamma > 0$ be any positive number that satisfies the equation

$$\gamma = \tan(\gamma). \tag{7}$$

Show that $X(x) = \sin(\gamma x)$ is an eigenfunction corresponding to the eigenvalue $\lambda = \gamma^2$.

(c) Using Green's second identity prove that eigenvalues are real and that the corresponding eigenfunctions can be chosen real.

G.4 Third test

Partial Differential Equations - AUC

Third exam - 25/11/2016

1. Let $u(x,t)$ a solution of the heat equation defined on the rectangle

$$R = \{(x,t) \mid 0 < x < 2, 0 < t < 3\}. \tag{1}$$

(a) Assume $u(x,t) \leqslant 1$ for all $(x,t) \in R$. If we know that $u(1,1) = 1$, what can we say about the value $u(1,2)$ and why ? (Use the *strong* maximum principle for $u(x,t)$.)

(b) Prove that the function $v(x,t) = u(x,t) + \epsilon x^2$ cannot have a local maximum in (the interior of) R. (This is part of the proof of the weak maximum principle for $u(x,t)$.)

2. Consider the following initial value problem for the heat equation on the real line

$$\begin{cases} u_t = u_{xx}, & -\infty < x < +\infty, t > 0, \\ u(x,0) = \frac{\sin x}{x}. \end{cases} \tag{2}$$

(a) Write the solution $u(x,t)$ using the Poisson integral formula. (You do not need to compute the integral).

(b) Show that the solution $u(x,t)$ is even in x at all times $t > 0$.

3. Solve the initial value problem for the heat equation on the interval with Dirichlet boundary conditions

$$\begin{cases} u_t = ku_{xx} & 0 \leqslant x \leqslant l, t \geqslant 0 \\ u(x,0) = \sin \frac{\pi x}{l}, \\ u(0,t) = 0, \\ u(l,t) = 0, \end{cases} \tag{3}$$

by looking for a separated solution of the form

$$u(x,t) = \sin\left(\frac{\pi x}{l}\right) T(t). \tag{4}$$

4. (a) Recall the definition of Fourier transform of a function $f(x)$.

(b) Compute the Fourier transform $\hat{f}(k)$ of the function

$$f(x) = \begin{cases} e^{-x} & x \geqslant 0, \\ 0 & x < 0. \end{cases} \tag{5}$$

(c) Using the inversion theorem for piecewise differentiable functions find the value of the inverse Fourier transform of $\hat{f}(k)$ at $x = 0$ and conclude that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+ik} dk = \pi. \tag{6}$$

5. (a) Let $\hat{f}(k)$ be the Fourier transform of the function $f(x)$. Prove the formula for the the Fourier transform of $f'(x)$.

(b) Solve the following initial value problem

$$\begin{cases} u_t = \frac{1}{2}u_{xx} + 4u & t > 0, x \in \mathbb{R}, \\ u(x,0) = \phi(x), \end{cases} \tag{7}$$

by applying the Fourier transform in the x variable, and solving the resulting ODE. Give the solution as a convolution of two functions.

Useful formula: the Fourier transform of the rescaled Gaussian function

$$\mathcal{F}[e^{-\frac{a}{2}x^2}] = \sqrt{\frac{2\pi}{a}} e^{-\frac{k^2}{2a}}, \quad a > 0. \tag{8}$$

G.5 Fourth test

Partial Differential Equations - AUC
Fourth exam - 21/12/2016

1. Let u be a harmonic function on an open bounded connected set $D \subset \mathbb{R}^2$ and let ∂D denote the boundary of D.
   (a) State the strong maximum and minimum principles for u.
   (b) Let v be an harmonic function on D as above. Show that if $|u(\mathbf{x}) - v(\mathbf{x})| \leqslant \epsilon$ for some $\epsilon > 0$ for all $\mathbf{x} \in \partial D$, then $|u(\mathbf{x}) - v(\mathbf{x})| \leqslant \epsilon$ for all $\mathbf{x} \in D \cup \partial D$.

2. Find the harmonic function $u(x, y)$ on the square

$$D = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < 1, 0 < y < 1\}$$

such that

$$u_x(0, y) = u_x(1, y) = u(x, 0) = 0, \quad u(x, 1) = cos(\pi x) + 2cos(3\pi x).$$

3. Consider the disk

$$D = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \,|\, |\mathbf{x}| < 2\}.$$

   (a) Let u be a harmonic function on D whose boundary value is

$$u_{|r=2} = 1 + \sin 5\theta - \cos 27\theta,$$

   where (r, θ) are the polar coordinates. What is $u(\mathbf{0})$?
   (b) Find the harmonic functions $u(r, \theta)$ on the D that are of the form

$$u(r, \theta) = R(r)\Theta(\theta) \tag{1}$$

   where (r, θ) are polar coordinates.
   (c) Using the Poisson formula find the value at the point $(r, \theta) = (1, 0)$ in D of the harmonic function u whose boundary value is $u_{|r=2} = \sin \phi$.

4. (a) If the Laplace transform of $f(t)$ is given by $F(s)$, derive the formula for the Laplace transform of $e^{bt} f(t)$.
   (b) The Laplace transform of t^k for k integer is $\frac{k!}{s^{k+1}}$. Use the formula derived in part (a) to obtain the Laplace transform of $e^{bt} t^k$.
   (c) Using the Laplace transform method, solve the initial value problem

$$y'' + 4y' + 4y = 6e^{-2t}t, \quad y(0) = 0, \quad y'(0) = 0.$$

5. Find the regions in the xy plane where the equation

$$yu_{xx} + 2u_{xy} + xu_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

H Solutions Tests 2016

H.1 First test

1. Using linearity we have
$$c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2) + \mathcal{L}(v) = g. \tag{1}$$
Setting $c_1 = c_2 = 0$ gives $\mathcal{L}(v) = g$, which implies
$$c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2) = 0. \tag{2}$$
Setting $c_1 = 1$, $c_2 = 0$, or $c_1 = 0$, $c_2 = 1$, we conclude.

2. (a) Since the ODE does not depend explicitly on y, let's reduce the order by introducing the variable $v = y'$. We get:
$$v' - \frac{2}{x}v - x^2 = 0, \tag{3}$$
which is a linear first order ODE with integrating factor
$$I = e^{-\int \frac{2}{x}\, dx} = e^{-2\log x} = x^{-2}. \tag{4}$$
Integrating the resulting exact equation we get the solution in implicit form
$$x^{-2}v - x = c_1, \tag{5}$$
therefore
$$v = x^3 + c_1x^2. \tag{6}$$
To get the solution in y we integrate this equation, getting
$$y = \frac{x^4}{4} + c_1\frac{x^3}{3} + c_2. \tag{7}$$
(b) Clearly the general solution is given by a particular solution to the inhomogeneous equation (simply setting $c_1 = c_2 = 0$), that is $\frac{x^4}{4}$, plus the general solution to the homogeneous equation, which is given by a linear combination of the two solutions 1 and $\frac{x^3}{3}$.
Computing the Wronskian
$$W(1, \frac{x^3}{3}) = \det\begin{pmatrix} 1 & \frac{x^3}{3} \\ 0 & x^2 \end{pmatrix} = x^2 \neq 0 \tag{8}$$
on the domain $x > 0$. Therefore the two solutions are linearly independent.

3. (a) Let $y = y(x)$ be a characteristic curve, then
$$0 = \frac{d}{dx}u(x, y(x)) = u_y(y' - (1 + x^2)), \tag{9}$$
therefore
$$y' = 1 + x^2, \tag{10}$$
which by integration gives the equation of the characteristic curves
$$y(x) = \frac{x^3}{3} + x + c. \tag{11}$$

(b)

$$u(x,y) = f(y - x - \frac{x^3}{3}) \tag{12}$$

(c) Just take the derivatives.

(d) Imposing the condition at $x = 0$ we get

$$u(0,y) = f(y) = e^{-\frac{y^2}{2}}, \tag{13}$$

therefore

$$u(x,y) = e^{-\frac{(y-x-\frac{x^3}{3})^2}{2}}. \tag{14}$$

(e) Looking for a solution $u(x,y) = g(x)$ one gets

$$g'(x) = \cos(x), \tag{15}$$

therefore $u(x,y) = \sin x$ is a particular solution of the inhomogeneous equation. By the superposition principle we get the general solution

$$u(x,y) = f(y - x - \frac{x^3}{3}) + \sin(x). \tag{16}$$

4. (a) Using the d'Alembert formula it's immediate to get

$$u(x,t) = \sin(x - 2t) + e^{x+2t}. \tag{17}$$

(b) The right-moving part is the function in $x - 2t$, while the left-moving the function in $x + 2t$.

H.2 Second test

1. (a) The even extension is defined as

$$\phi_e(x) = \begin{cases} \phi(x) & x > 0, \\ \phi(-x) & x < 0, \end{cases} \tag{18}$$

and similarly for ψ. By a simple change of variable $x \to -x$ in the d'Alembert formula

$$u(x,t) = \frac{1}{2}(\phi_e(x+ct) + \phi_e(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_e(\zeta)d\zeta \tag{19}$$

one finds that $u(-x,t) = u(x,t)$, hence u is even in x for all t. The derivative of an even function is an odd function. Indeed taking the derivative in x of $u(-x,t) = u(x,t)$ one gets $u_x(-x,t) = -u_x(x,t)$. So it vanishes at $x = 0$ and satisfies the Neumann boundary condition. Clearly $u(x,t)$ solves the wave equation and the initial conditions on the positive real line, so it solves the i.v.p. (1).

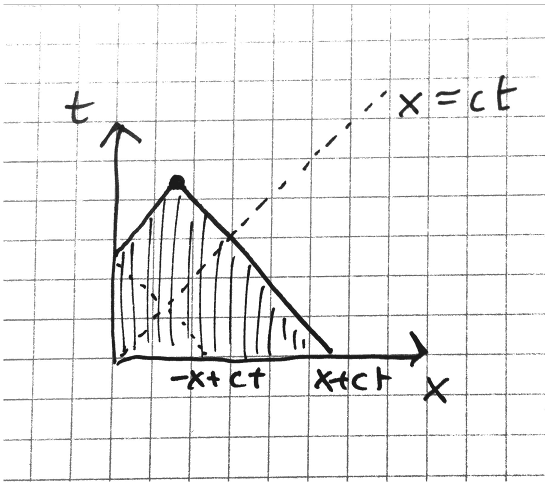
(b) For $x > ct$ we have both $x - ct > 0$ and $x + ct > 0$, so the even extensions ϕ_e, ψ_e appearing in d'Alembert formula can be just replaced with the corresponding ϕ, ψ, giving

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\zeta)d\zeta. \tag{20}$$

For $x < ct$ we have $x + ct > 0$ but $x - ct > 0$, so we have to split the integral in two parts, one with $\zeta > 0$ and one with $\zeta < 0$, which gives

$$u(x,t) = \frac{1}{2}(\phi(x+ct)+\phi(-x+ct))+\frac{1}{2c}\int_0^{x+ct} \psi(\zeta)d\zeta+\frac{1}{2c}\int_0^{-x+ct} \psi(\zeta)d\zeta. \tag{21}$$

(c) Sketch:



2. (a) Recall the definition of energy

$$E = \int_0^l (\frac{1}{2}\rho u_t^2 + \frac{1}{2}T u_x^2)dx. \tag{22}$$

Taking the t derivative we get

$$\frac{dE}{dt} = \int_0^l (\rho u_t u_{tt} + T u_x u_{xt})dx. \tag{23}$$

Using the wave equation the integrand becomes

$$T(u_t u_x)_x, \tag{24}$$

therefore

$$\frac{dE}{dt} = T(u_t u_x)|_0^l \tag{25}$$

which vanishes because of the boundary conditions.

(b) The energy at $t = 0$ is zero, so it has to be zero at all times because of energy conservation. But if $E = 0$ also the integrand has to be zero, because it is the sum of two positive functions. Therefore $u_x = u_t = 0$ for all x and t, so $u(x,t)$ is constant, and equal to zero.

3. (a) This is solved in the book, section 4.1

   (b) A linear combination of the separated solutions gives the general solution of the b.v.p.

$$u(x,t) = \sum_{n\geqslant 1} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{n\pi t}{l} + B_n \sin\frac{n\pi t}{l}\right). \tag{26}$$

   Setting $t = 0$ in this general solution, and in its t derivative, we get sine Fourier series for the initial conditions

$$\sin\frac{\pi x}{l} = \sum_{n\geqslant 1} A_n \sin\frac{n\pi x}{l}, \tag{27}$$

$$\frac{2\pi}{l}\sin\frac{2\pi x}{l} = \sum_{n\geqslant 1} B_n \frac{n\pi}{l}\sin\frac{n\pi x}{l}. \tag{28}$$

   Using orthogonality formulas we find A_n is 1 for $n = 1$ and is 0 otherwise; B_n is 1 for $n = 2$ and zero otherwise. The general solution is therefore

$$u(x,t) = \sin\frac{\pi x}{l}\cos\frac{\pi t}{l} + \sin\frac{2\pi x}{l}\cos\frac{2\pi t}{l}. \tag{29}$$

4. (a) For $\lambda = 0$ the e.v. equation becomes $X'' = 0$, so has general solution $X = Ax + B$. Imposing the boundary conditions we get that $B = 0$, so the eigenfunction is $X = x$.

   (b) Computing, for $X = \sin\gamma x$:

$$-X'' = -\gamma^2 \sin\gamma x = -\gamma^2 X = -\lambda X. \tag{30}$$

   We need to check the b.c.: $X(0) = \sin 0 = 0$ and

$$X'(1) = \gamma\cos\gamma = \tan\gamma\cos\gamma = \sin\gamma = X(1). \tag{31}$$

   So, X is indeed an eigenfunction with eigenvalue $\lambda = \gamma^2$.

   (c) Let X_1, X_2 be eigenfunctions corresponding with eigenvalues λ_1, λ_2. Then the second Green's identity says that

$$(\lambda_1 - \lambda_2)\int_0^1 X_1 X_2 dx = (-X_1' X_2 + X_1 X_2')|_0^1. \tag{32}$$

   The righthand side vanishes because of the b.c., hence $(X_1, X_2) = \int_0^1 X_1 X_2 dx$ is zero, i.e., the eigenfunctions are orthogonal.

H.3 Third test

1. (a) By the strong maximum principle a solution of the heat equation cannot assume its maximum value in the interior of R, unless it is constant. In this case u assumes the maximum value at $(1, 1)$ so it is constant and equal to 1. Therefore $u(1, 2) = 1$.

(b) The function $v(x,t)$ satisfies the diffusion inequality

$$v_t - kv_{xx} = -2\epsilon k < 0. \tag{1}$$

If $v(x,t)$ had a local maximum at the point (x,t) in the interior of R, then at that point $v_t = 0$ and $v_{xx} \leqslant 0$, and we would get a contradiction with the diffusion inequality.

2. (a)

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \frac{\sin y}{y} dy. \tag{2}$$

(b)

$$\begin{aligned} u(-x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(-x-y)^2}{4t}} \frac{\sin y}{y} dy & (3) \\ &= -\frac{1}{\sqrt{4\pi t}} \int_{+\infty}^{-\infty} e^{-\frac{(-x+y)^2}{4t}} \frac{\sin(-y)}{-y} dy & (4) \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \frac{\sin y}{y} dy & (5) \\ &= u(x,t). & (6) \end{aligned}$$

3. Substituting the separated solution in the heat equation we get

$$\sin\left(\frac{\pi x}{l}\right) T'(t) = -k\left(\frac{\pi}{l}\right)^2 \sin\left(\frac{\pi x}{l}\right) T(t), \tag{7}$$

therefore $T(t)$ satisfies

$$T'(t) = -k\left(\frac{\pi}{l}\right)^2 T(t), \tag{8}$$

which is solved by

$$T(t) = Ce^{-k\left(\frac{\pi}{l}\right)^2 t}. \tag{9}$$

Taking into consideration the initial condition we have that $C = 1$, hence the solution is

$$u(x,t) = \sin\left(\frac{\pi x}{l}\right) e^{-k\left(\frac{\pi}{l}\right)^2 t}. \tag{10}$$

4. (a)

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx. \tag{11}$$

(b) Integrating an exponential we get

$$\hat{f}(k) = \frac{1}{1+ik}. \tag{12}$$

(c) The inversion theorem for piecewise differentiable functions in this case says that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ikx} \frac{1}{1+ik} dk = \frac{1}{2}(f(x_+) + f(x_-)), \tag{13}$$

that for $x = 0$ gives the result.

5. (a) Differentiating w.r.t. x the inversion formula $f(x) = \mathcal{F}^{-1}[\hat{f}(k)]$ we get the desired result $ik\hat{f}(k)$.

(b) Taking the Fourier transform of the PDE we get

$$\hat{u}_t = (-\frac{1}{2}k^2 + 4)\hat{u}, \tag{14}$$

which has solution

$$\hat{u}(k,t) = c(k)e^{(4-\frac{1}{2}k^2)t}. \tag{15}$$

The initial condition at $t = 0$ fixes $c(k) = \hat{\phi}(k)$. Applying the inverse Fourier transform gives

$$u(x,t) = \mathcal{F}^{-1}[\hat{\phi}(k)e^{(4-\frac{1}{2}k^2)t}] = \phi * \mathcal{F}^{-1}[e^{(4-\frac{1}{2}k^2)t}]. \tag{16}$$

The formula for the Fourier transform of the Gaussian function gives

$$\mathcal{F}^{-1}[e^{(4-\frac{1}{2}k^2)t}] = \frac{1}{\sqrt{2\pi t}}e^{4t-\frac{1}{2}x^2}, \tag{17}$$

therefore

$$u(x,t) = \frac{e^{4t}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2t}}\phi(y)dy. \tag{18}$$

H.4 Fourth test

1. (a) If u is not constant, the maximum (resp. minimum) of u is achieved only on ∂D.

   (b) We have that
   $$-\epsilon \leqslant u(\mathbf{x}) - v(\mathbf{x}) \leqslant \epsilon \tag{19}$$
   for all $\mathbf{x} \in \partial D$. Since both maximum M and minimum m of $w = u - v$ appear on ∂D, we have that $M \leqslant \epsilon$ and $-\epsilon \leqslant m$. By definition of maximum and minimum on D we conclude
   $$-\epsilon \leqslant m \leqslant u(\mathbf{x}) - v(\mathbf{x}) \leqslant M \leqslant \epsilon \tag{20}$$
   on D.

2. As usual, perform separation of variables, using only the homogeneous boundary conditions, and consider separately the cases of positive, negative, and zero eigenvalue. The linear combination of all separated solutions with arbitrary coefficients gives
   $$u(x, y) = \frac{A_0}{2} y + \sum_{n \geqslant 1} A_n \cos(n\pi x) \sinh(n\pi y). \tag{21}$$
   Imposing the remaining boundary condition allows to easily fix the coefficients, getting
   $$u(x, y) = \frac{\cos(\pi x) \sinh(\pi y)}{\sinh(\pi)} + \frac{2\cos(3\pi x) \sinh(3\pi y)}{\sinh(3\pi)}. \tag{22}$$

3. (a) The average of $\sin n\theta$ and $\cos n\theta$ for any integer n is zero. Therefore the value of u at the origin is 1.

   (b) See textbook p.165-6.

   (c) Substituting in Poisson formula we get
   $$u(1, 0) = \frac{3}{2\pi} \int_0^{2\pi} \frac{\sin\phi}{5 - 4\cos\phi} d\phi, \tag{23}$$
   which equals zero (e.g., because we can integrate on the symmetric interval $(-\pi, \pi)$ and the integrand is odd).

4. (a)
   $$F(s - b) \tag{24}$$
   (b)
   $$\frac{k!}{(s - b)^{k+1}} \tag{25}$$
   (c) Taking Laplace transform we get
   $$Y = \frac{6}{(s + 2)^4}, \tag{26}$$
   therefore the solution is
   $$y = t^3 e^{-2t}. \tag{27}$$

5. It is parabolic on the hyperbola $xy = 1$. It is hyperbolic in the region containing the origin, and elliptic in the other two regions.

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